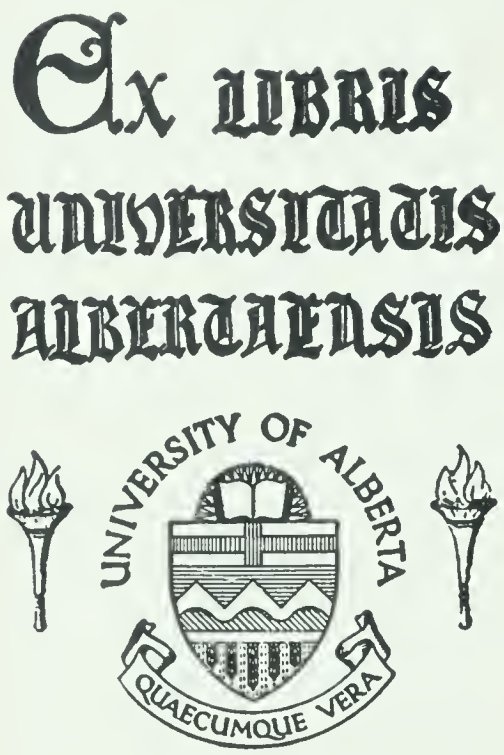
For Reference

NOT TO BE TAKEN FROM THIS ROOM

For Reference

NOT TO BE TAKEN FROM THIS ROOM



University of Alberta
Printing Department

Digitized by the Internet Archive
in 2019 with funding from
University of Alberta Libraries

https://archive.org/details/ultrasonicattenu00regi

ULTRASONIC ATTENUATION

IN SOLIDS.

Thesis
1960
#1D

THE UNIVERSITY OF ALBERTA

ULTRASONIC ATTENUATION AND DISPERSION IN NORMAL METALS AT LOW TEMPERATURES AND IN POLYCRYSTALS.

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF DOCTOR OF PHILOSOPHY.

DEPARTMENT OF PHYSICS

BY

REGINALD ALBERT MOORE, B.Sc., M.Sc.

EDMONTON, ALBERTA

APRIL, 1960.

UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "Ultrasonic Attenuation and Dispersion in Normal Metals at Low Temperatures and in Polycrystals", submitted by Reginald Albert Moore in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

DOCTOR OF PHILOSOPHY (1960)
(Physics)

UNIVERSITY OF ALBERTA
Edmonton, Alberta.

TITLE: Ultrasonic Attenuation and Dispersion in Normal Metals at Low Temperatures and in Polycrystals.

AUTHOR: Reginald Albert Moore; B.Sc., M.Sc.

SUPERVISOR: Dr. A. B. Bhatia.

SCOPE AND CONTENTS:

PART I: ULTRASONIC ATTENUATION IN NORMAL METALS AT LOW TEMPERATURES.

Below about 20° K the attenuation of sound waves in a single metal crystal arises primarily from the electron-ion interaction. When a sound wave passes through a metal the ions are displaced from their equilibrium positions and an electric field is set up in the metal. This field exerts a force on the ions; in addition there is a force due to the transfer of momentum to the ions in electron-ion collisions. Because of the finite relaxation time for the electrons, the resultant force is out of phase with ionic motion and hence leads to attenuation and dispersion of sound waves. The problem is formulated in terms of the Boltzmann transport equation. When the collision integral in the Boltzmann equation is characterized by a single relaxation time our procedure gives the same results as obtained by previous authors.

Next a method for solving the Boltzmann equation at low temperatures is given when the collision integral is due to the interaction of the electrons with phonons and impurities. Although, in principle, the method is applicable for all relevant wave-lengths, Λ, of the sound waves, explicit solutions have been obtained only for the limiting cases $\Lambda \gg \ell$ and $\Lambda \ll \ell$, where ℓ is the mean free path

of the electrons. For $\Lambda >> \ell$ it is found that the relaxation time, τ_a , appropriate for attenuation is the same for both shear and dilational waves. However, τ_a is different from τ_c , the relaxation time appropriate for electrical conductivity, σ . Further, it is found that the relaxation time appearing in the expression for the attenuation for extremely high frequency shear waves is to be interpreted as τ_c . Lastly, it is shown that where electron-electron collisions are important their relaxation time may be deduced from the observed temperature variation of the attenuation and σ.

PART II: ULTRASONIC ATTENUATION IN POLYCRYSTALS.

In a polycrystal the crystallographic axes of the different grains are differently oriented with respect to a fixed coordinate system. Since the effective elastic constants of a grain depend upon its orientation, they differ from grain to grain and hence the sound waves are scattered. In this part of the thesis we calculate the scattering (and hence the attenuation) of plane shear and dilational waves due to the random orientation of the grains in a polycrystal. These calculations are a generalization of the work of earlier authors to take into account the fact that an elastically anisotropic medium has actually several independent elastic constants. The final expressions for the attenuation coefficients exhibit qualitatively similar features as the corresponding expressions of previous work but differ quantitatively from them.

ACKNOWLEDGEMENTS:

First I would like to thank Dr. A. B. Bhatia for introducing the problems to me and for his steadying guidance and instructional assistance throughout the course of my work.

I also wish to thank the Ontario Research Foundation for their generous financial aid, in the form of fellowships and summer grants, enabling me to work on these problems.

Finally, I would like to thank the International Nickel Company for supplying me with a Teaching Fellowship during my final year here.

TABLE OF CONTENTS:

	Page
General Introduction	1
PART I. Ultrasonic Attenuation and Dispersion in Normal Metals at Low Temperatures	8
1. Introduction	9
1.1 Resumé of Earlier Theoretical Work	9
1.2 Outline of the Work in this Thesis	15
2. Calculation of the Attenuation and the Dispersion Relations Assuming the Existence of a Single Relaxation Time	20
2.1 Formulation	20
2.2 Incident Plane Dilational Waves	26
2.3 Incident Plane Shear Waves	31
2.4 Discussion	36
3. Calculation of the Relaxation Times for the Electron-Phonon Interaction	39
3.1 Formulation	39
3.2 The Relaxation Times, τ_ρ.	54
3.3 Low Frequency Incident Plane Shear Waves	61
3.4 Low Frequency Incident Plane Dilational Waves	64
3.5 High Frequency Incident Plane Shear Waves	68
3.6 High Frequency Incident Plane Dilational Waves ...	76
4. Calculation of the Relaxation Times for Impurity Scattering	81
5. Discussion	86

Page

PART II. Ultrasonic Attenuation in Polycrystals 92

6. Introduction 93

7. Formulation and Solution for the Scattered Wave from a Single Inhomogeneity 97

7.1 The Equations of Motion for the Bulk Material 98

7.2 The Equations of Motion for the Inhomogeneity ... 100

7.3 Solutions for the Scattered Wave 103

8. The Attenuation Coefficient 109

8.1 The Scattering Cross-Section for the Single Inhomogeneity 109

8.2 The Attenuation Coefficient for a Polycrystal ... 112

9. Discussion 116

Appendix I 120

Bibliography 134

LIST OF TABLES:

Page

Table I Values of $\left(\frac{T}{\Theta}\right)^{n-5} \frac{J_n}{J_5}$ for n = 5,7,...,17, $\frac{T}{\Theta}$ = 0.05... 60

Table II Values of D_{21}/T, D_{25}/T, and D_{25}/D_{21} for aluminum and magnesium 119

LIST OF FIGURES:

		Page
Figure 1	Dependence of Ultrasonic Attenuation in Lead on Temperature	2
Figure 2	Dependence of Ultrasonic Attenuation in Lead on Temperature *Below 12°K*	4
Figure 3	Block Diagram of the Experimental Apparatus Used to Study Sound Waves in Solids	7
Figure 4	Plot of τ_1^i/τ_2^i versus $\beta = 1 + \mu^2/2K_o^2$	85
Figure 5	Plot of $[\alpha_\ell(\omega)/\omega]\left[\lim_{\omega\to\infty}\{\alpha_\ell(\omega)/\omega\}\right]^{-1}$ versus $k\ell$, ...	91
Figure 6	The Euler Angles, α, β, and γ	122

GENERAL INTRODUCTION

A plane shear or a plane dilational sound wave propagating in solids suffers attenuation. The attenuation, α, is defined as the reciprocal of the distance in which the intensity of the wave is reduced to $1/e$ of its value.

In the case of metals, whose linear dimensions are large compared with the wavelength, Λ of the sound wave, the attenuation varies roughly with temperature as in figure 1. The attenuation at room temperatures and higher and the maximum around $100^{\circ}K$ are due to the interaction of the sound wave with dislocations. The precise temperature at which this maximum occurs depends on the angular frequency, ω, of the sound wave and on the crystal. The theories of this interaction have been developed by Koehler, (1952) [1], and Granato and Lücke, (1956) [2], for the room temperature region and by Seeger, (1956) [3], for the so-called Bordoni peak at intermediate temperatures. We shall not be concerned with these portions of the temperature range in this thesis, but shall deal with the attenuation below about $20^{\circ}K$. Here again, the precise temperature dependence of the attenuation depends upon the frequency of the sound wave and on the crystal.

The mechanism producing the attenuation below about $20^{\circ}K$ is largely the interaction of the sound wave with the electrons, as first concluded by Dömmel and Mason, (1956) [4]. The main evidence for this conclusion is two-fold: (1) In normal metals and for sound waves with wavelengths much greater than the mean free path of the electrons the temperature dependence of α is qualitatively the same as that for the electrical conductivity. This shows that the same interaction is,

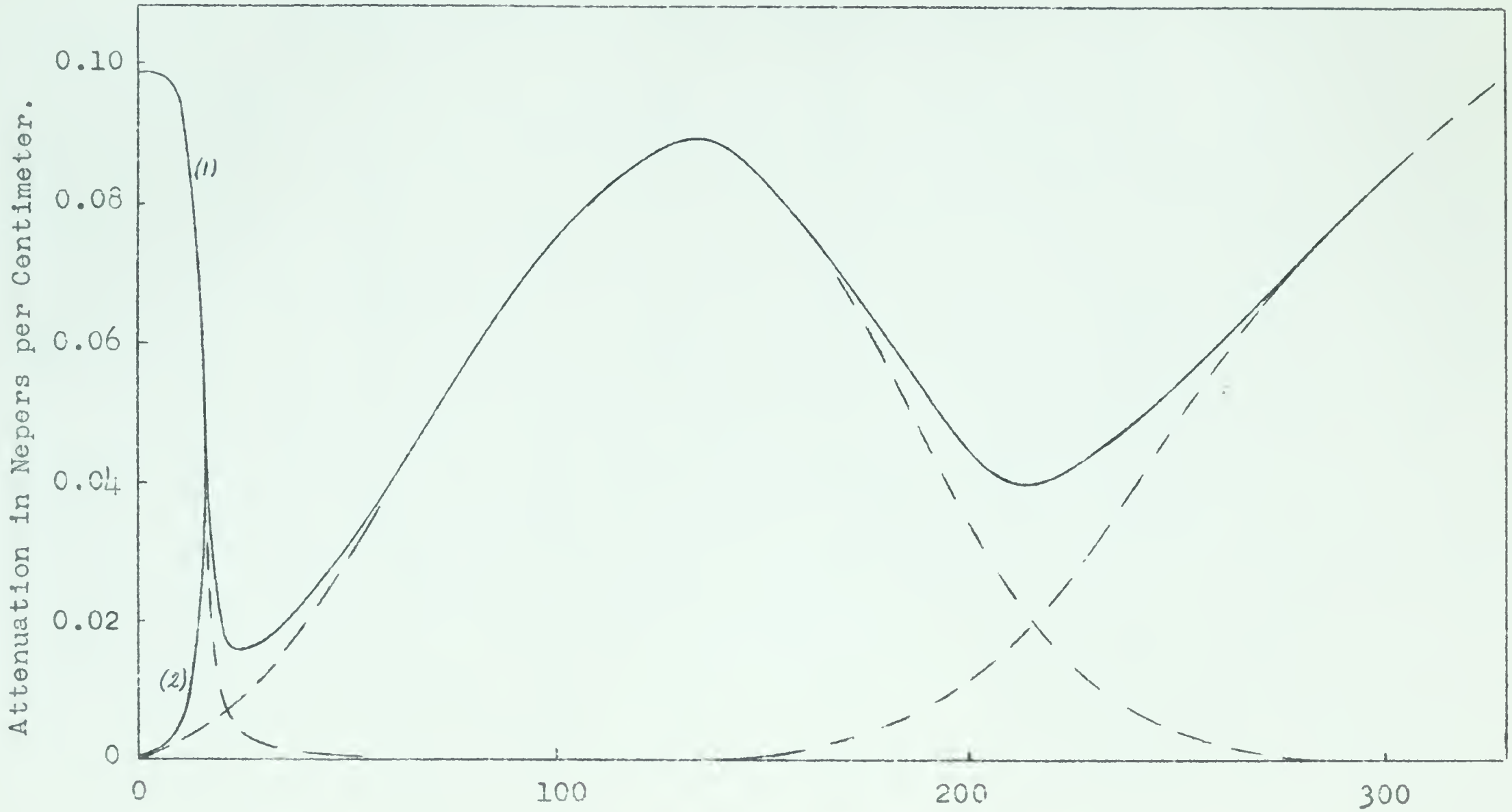


Figure 1.

Dependence of Ultrasonic Attenuation in Lead on Temperature,
(1) For the Normal State of the Metal, and (2) For the Superconducting State.

to a large extent, responsible for both of these effects. (2) In superconducting materials the attenuation drops sharply at the transition temperature to a very low value and decreases rapidly to zero as the temperature approaches absolute zero. Both shear and dilational sound waves behave similarly and a typical plot is illustrated in Figure 2.

Part I of the present thesis is a contribution to the theoretical study of the attenuation of sound waves in normal metals resulting from their interactions with the electrons in the metal. The relationship of our work to that of previous authors will be discussed in the introduction to Part I.

In the case of polycrystalline materials there are additional mechanisms causing attenuation. One of these arises from the fact that the crystallographic axes of the different grains in the polycrystal are differently oriented with respect to a set of coordinate axes fixed in space, that is, the laboratory coordinate system. As a result the effective elastic constants determining the propagation of the sound wave in a given direction varies from grain to grain and this gives rise to a scattering of the sound waves. When the linear dimensions of the grains are small compared to the wavelength of the sound waves, the attenuation arising from this scattering process is given by

$$\alpha_g = Df^4, \qquad (1)$$

where D is a constant depending upon the specimen used and its elastic constants and $f = \omega/2\pi$ is the frequency of the sound wave. The variation of the constant D with temperature arises from the variation of the elastic constants with temperature and over a range of $0^{\circ}K$ to $300^{\circ}K$

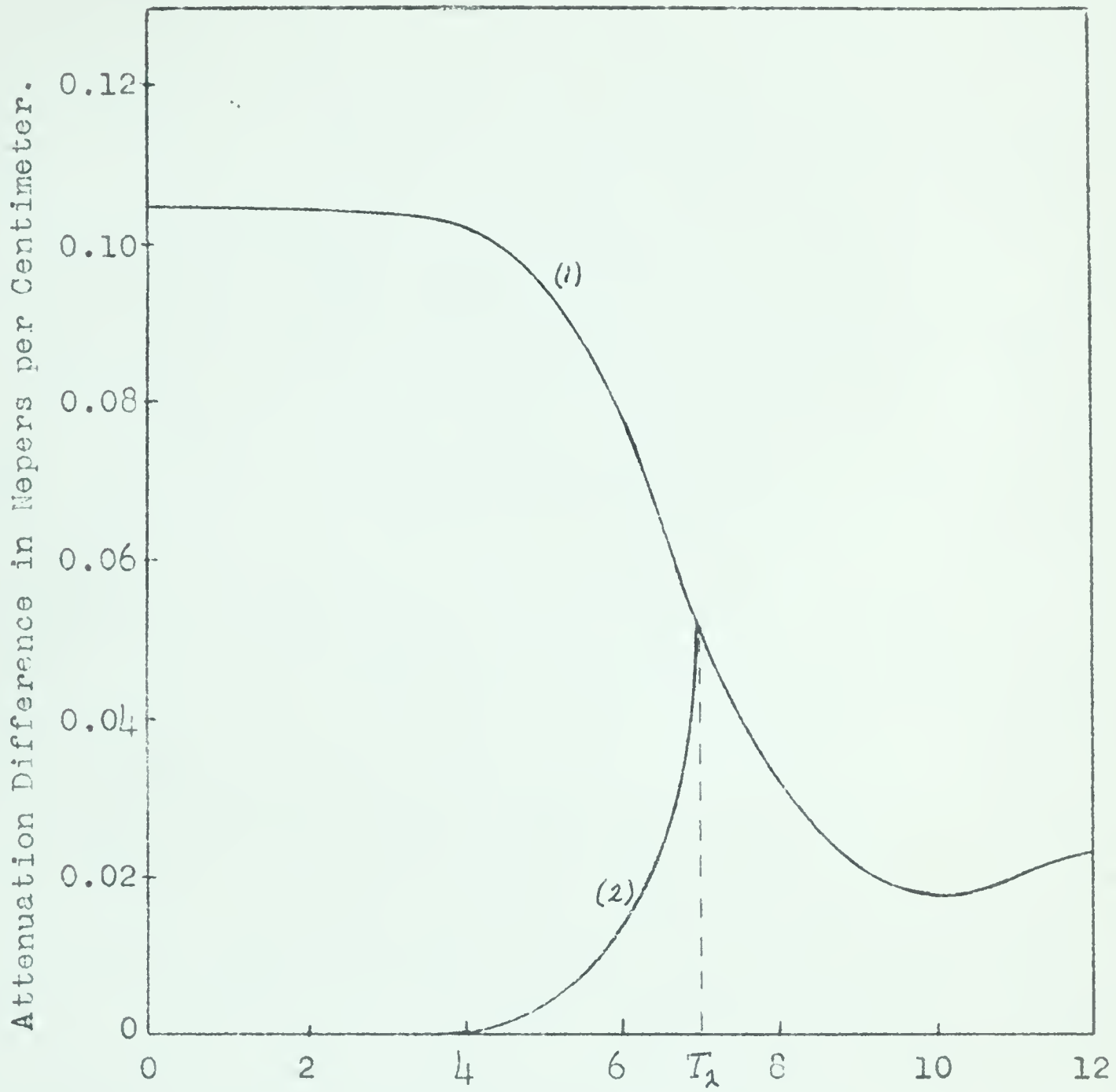


Figure 2.

Dependence of Ultrasonic Attenuation in Lead on Temperature Below 12°K.

(1) For the Normal State of the Metal, and (2) for the Superconducting State. The Normal State is Retained Below the Transition Temperature, T_λ, by the Application of a Magnetic Field.

is not more than 30%. In view of the facts that the other mechanisms, some of them being the same as those responsible for attenuation in single crystals, depend sensitively on temperature and have a frequency dependence which is substantially different from a fourth power law, it is not difficult to isolate the attenuation arising from the scattering due to random orientations of the grains. Part II of this thesis is a calculation of D in (i). Again the relationship of our work with that of previous authors will be discussed in the introduction to Part II.

It would be useful to note the frequency ranges and wavelengths of the sound waves for which the mechanisms studied here contribute to attenuation. For Part I the first measurements were made in the megacycle region, but very recently Lax, (1959) [5], has extended the lower limit to close to ten kilocycles per second. The upper limit to the observable frequency range is determined by the experimental difficulties in obtaining very high frequency sound waves of appreciable energies. In Part II the fourth power dependence becomes measurable in the megacycle range. Taking an average velocity of 3×10^5 centimeters per second for the sound wave, frequencies of 10^4 cycles per second represent wavelengths of about 30 centimeters and frequencies of 10^8 cycles per second represent wavelengths of about 0.003 centimeters.

Since the experimental procedure is outside the scope of this thesis, it will be mentioned only briefly. The usual method of generating ultrasonic waves in solids is that of piezoelectric and magnetostrictive transducers, such as quartz. In the megacycle range the pulse technique is used. This technique consists of sending ultrasonic pulses, short with respect to the length of the specimen but of about one hundred

wavelengths so as to ensure steady state conditions throughout most of the pulse, through the sample at regular intervals. The velocity and attenuation of the initial pulse or the reflected pulse, (echo-technique), are measured. A block diagram of the experimental apparatus is shown in figure 3. At lower frequencies the pulse technique cannot, of course, be used, so that here the velocity and attenuation are measured by setting up standing waves in the sample and measuring their time rate of decay.

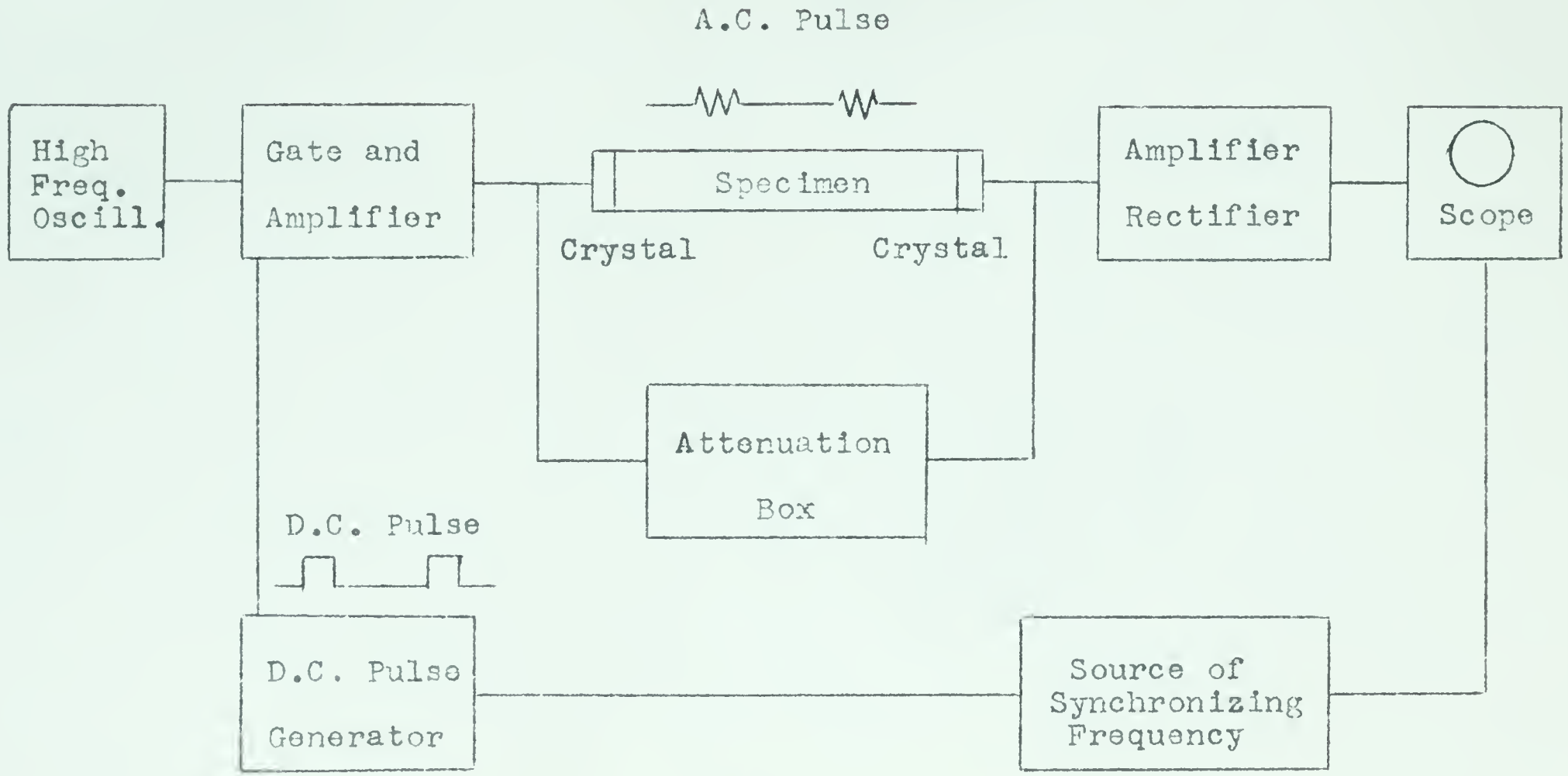


Figure 3.

Block Diagram of the Experimental Apparatus Used to Study Sound Waves in Solids.

PART I

ULTRASONIC ATTENUATION AND DISPERSION IN NORMAL METALS AT LOW TEMPERATURES.

1. INTRODUCTION

As already mentioned, the attenuation of sound waves in metals at liquid helium temperatures arises from the interaction between the conduction electrons and the ions in the metal. In this section we shall review briefly the earlier theoretical work and then outline the work done in this thesis.

1.1 REVIEW OF EARLIER THEORETICAL WORK

The first ~~theoretical~~ work on the subject is due to Mason and Bömmel, (1954) [18], and Bömmel, (1956) [4], who ascribed the attenuation to be due to the viscosity of the electron gas in the metal. If the electron gas is assumed to be rigidly coupled to the ions, the viscous force per unit area on an element of volume of the metal contributes to the shearing stress a term of the type

$$X_y = T_{12} = \mu S_{12} + \eta \dot{S}_{12}, \tag{1}$$

where S_{12} is the shearing strain, $\dot{S}_{12}$ is its time rate of change, η is the coefficient of viscosity, and μ is the shear elastic constant. The attenuation coefficient, α, is then obtained in the usual way (See for example, Mason, Electromechanical Transducers and Wave Filters), resulting in

$$\alpha = \frac{\eta \omega^2}{2\rho V_s^3} \tag{2}$$

for shear waves and

$$\alpha = \frac{\omega^2}{2\rho V_l^3}\left(\frac{4}{3}\eta + \chi\right) \tag{3}$$

for dilational waves. χ is the compressional viscosity, V_s and V_ℓ are the velocities of the shear waves and of the dilational waves, respectively, in the metal. From the experimental evidence on α for the two types of waves, Mason and Bömmel, [4], have concluded that $\frac{\chi}{\eta} << 1$ for the electron gas.

The kinetic theory expression for the viscosity of a gas is

$$\eta = \frac{1}{3} Nm\ell\bar{v}, \tag{4}$$

where N is the number of particles per unit volume, m is the mass of a particle, ℓ is the mean free path, and $\bar{v}$ is the mean velocity. For electrons we have

$$\frac{1}{2} m\bar{v}^2 = \frac{3}{10} \frac{\hbar^2}{m} \left(3\pi^2 N\right)^{2/3}, \tag{5}$$

where N is now the number of free electrons per unit volume. We note that expressions (2) and (3) for attenuation are proportional to the square of the frequency. The temperature dependence of the attenuation comes about through the dependence of ℓ on temperature. Mason and Bömmel assumed that the mean free path appearing in (4) may be taken to be the same as that which appears in the expression for the electrical conductivity, σ, when written in the form

$$\sigma = \frac{Ne^2}{m} \frac{\ell}{\bar{v}} \quad \text{or} \quad \ell = \frac{\sigma m\bar{v}}{Ne^2}. \tag{6}$$

Thus the attenuation would be proportional to the electrical conductivity.

A treatment such as above, in terms of the viscosity of the electron gas, is valid only as long as the mean free path is small com-

pared to the wavelength, Λ, of the sound wave, that is, for $k\ell < 1$ where $k = 2\pi/\Lambda$. When $\ell > \Lambda$ it is no longer permissible to isolate for consideration a portion of the metal small compared to Λ. Moreover, the passage of a sound wave through the metal produces an electric field in the metal and, therefore, in the presence of the sound wave the electrons are affected both by collisions with the ions and by the electric field. A comprehensive treatment which takes both these effects into account has been given by Pippard, (1955) [6], in terms of the free electron model of a metal.

The basic ideas underlying his work are as follows. When the lattice ions are displaced from their equilibrium positions due to an ultrasonic wave passing through a metal, an electric field is set up inside the metal. This electric field and the stationary distribution function of the electrons are determined by the collisions of the electrons with the lattice vibrations and other defects in the metal, by the equation of continuity, and by Maxwell's electromagnetic equations. The resultant distribution function is not identical with that which the electrons would have if they were locally in thermal equilibrium. Hence there is a dissipation of sound energy into heat energy. Pippard adopted the kinetic approach to the problem, namely, that of following a single electron at the Fermi surface and considering the changes in its velocity due to both the electric field generated by the sound waves and the collisions. For the latter he assumed that the velocity relaxes exponentially to its value corresponding to local equilibrium with a relaxation time, τ. The attenuation of an ultrasonic wave is then obtained by calculating the Joule heat associated with the steady state electronic

current set up by the wave.

Using the same basic ideas as those used by Pippard, Steinberg (1958a) [7], has formulated the problem in terms of the Maxwell-Boltzmann transport equation. In this work the collision integral is simplified by the assumption that, due to the collisions, the distribution function, f, for the electrons relaxes towards a distribution function $\bar{f}$, which the electrons would have if they were locally in thermal equilibrium, with a relaxation time, τ , so that

$$\left[\frac{\partial f}{\partial t}\right]_{coll.} = - \frac{(f - \bar{f})}{\tau} , \qquad (7)$$

Steinberg then sets up the equations of motion for the ions by adding to the elastic stress tensor a kinetic stress tensor associated with the steady state distribution function of the electrons. The components of this kinetic stress tensor are given by

$$T_{xy} = - \left(\frac{2m^4}{h^3}\right) \int v_x v_y \left[f(\vec{v}) - \bar{f}(\vec{v}) \right] d\vec{v} . \qquad (8)$$

This procedure is somewhat analogous to that used in the kinetic theory of gases in order to obtain the Navier-Stokes equation of hydrodynamics from the Maxwell-Boltzmann transport equation. By solving his equations of motion for plane waves, Steinberg is able to calculate the dependence of both the attenuation and the sound velocity on frequency. The analysis is carried out for both shear and dilational waves and the results on attenuation are in agreement with those obtained by Pippard. However, Steinberg's own conclusion that his results are at variance with Pippard's is, in our opinion, based on an erroneous argument and we discuss this in section 2.4 of this thesis.

Independently of Pippard and Steinberg, Kleiner et al, (1957) [8], have also made a substantially similar Boltzmann formulation of the problem. Their analysis consists of a calculation of the attenuation for dilational waves in the limits of very low and very high frequencies by calculating the entropy change in the electron gas.

In the works described above the various authors have assumed the existence of a relaxation time for the distribution function of the electrons. Even if a relaxation time for the attenuation problem does exist, it is not evident that this relaxation time should be the same as that appearing in the expression for electrical conductivity as is assumed by Mason and Pömmel. Nor, as pointed out by Steinberg, (1958), is there any "à priori" reason to assume that the relaxation times appearing in the expressions for the attenuation of dilational and shear waves are the same. There is also the question as to whether the same relaxation time is appropriate to describe the attenuation at both very low and very high frequencies in the case of shear waves.

Before outlining the work done in the present thesis, we note that Steinberg, (1958b) [9], has calculated the viscosity of a degenerate electron gas. He finds that at low temperatures the relaxation time resulting from the electron-phonon interaction, appropriate for the viscosity, is approximately one-third the relaxation time appropriate for electrical conductivity. We shall designate this latter relaxation time by τ, hereafter. For impurity scattering one obtains a relaxation time $\underline{a}$ times τ', the residual relaxation time for electrical conductivity, where $\underline{a}$ is a constant of the order of, but less than,

unity. These results are relevant for the attenuation of low frequency, $k\ell_1 << 1, \omega\tau_1 << 1$, shear waves.

1.2 OUTLINE OF THE WORK IN THIS PAPER.

In the present work we carry out several progressive calculations. These calculations, in common with those described previously use the Sommerfeld free electron model of a metal. The generalization to a model where the electrons are described by a single effective mass throughout the band is straightforward. We also use in our calculations Pippard's basic idea that the electric field produced by the passage of a sound wave and the stationary distribution function of the electron are determined by the various kinds of collisions the electrons can suffer, by the equation of continuity, and by Maxwell's electromagnetic equations.

In section 2.1 we formulate the problem in terms of the Maxwell-Boltzmann transport equation. The first step is to assume that the collision integral can be characterized by a unique relaxation time, τ, in accordance with (7). We are now able to determine the electric field and the stationary distribution function, f, for the electrons in terms of τ. This part of the work is, of course, identical with Steinberg's procedure.* We next set up the equations of motion for wave propagation in a metal. Instead of calculating the kinetic stress tensor associated with f as Steinberg has done, we use the following physical ideas. The electric field set up in the metal due to the passage of

---

*Our work was done independently of Steinberg's and a summary was sent to the "International Conference on the Transport Properties of Metals at Low Temperatures", Geneva, New York, 1958. Some of the results on the calculation of the relaxation times contained in this part of this thesis were reported at the January, 1960, meeting of the American Physical Society, Bulletin of the American Physical Society 1960 Annual Meeting at New York, C12, Page 14.

the ultrasonic wave exerts a force on the positively charged ions, the electric force. Further, in electron-ion collisions there is a direct transfer of momentum from the electrons to the ions per unit time and this results in a force on the ions, the impulsive force. Finally there is the elastic restoring force between the ions independent of the electron-ion interaction, that is, even when the impulsive force and the electric force are zero. The resultant of these three forces is, because of the finite relaxation time for the electrons, in general, not in phase with the displacements of the ions and hence the waves are attenuated. In this manner we are able to calculate both the attenuation and the dispersion of the sound waves.

The expressions for the attenuation are identical with those obtained by Pippard and Steinberg. However, our procedure of setting up the equations of motion for the metal gives us an interesting insight into the significance of τ in one particular case. It is found that the attenuation of a shear wave with an angular frequency such that $\omega\tau >> 1$ results solely from the impulsive force. Since this impulsive force arises from the direct transfer of momentum from the electrons to the ions, it will contain the same relaxation time as that appearing in the expression for the electrical conductivity. Hence, in this high frequency limit, the attenuation will also contain this relaxation time. A confirmation of this expectation is obtained in section 3.5.

As regards to the dispersion, small *changes in the velocity of the sound waves* [~~velocity change~~], of the order of m/M, the mass of the electron over the mass of the ion, are associated with the attenuation. Incidentally, our method also yield an expression for the contribution to the velocity of the sili-

tional waves arising from the variation in the electronic charge density associated with the passage of this type of wave in a metal. This contribution is frequency dependent for high frequency sound waves ($\omega > 10^{10}$ radians per second), but is independent of the relaxation time for the electrons. We discuss this contribution further in section 2.4.

In sections 3 and 4 we discard the assumption of the existence of a single relaxation time and consider the solutions of the transport equation when the collision integral is due to the electron-phonon interaction, (section 3), and to the electron-impurity scattering (section 4). For the electron-phonon interaction we start with the collision integral as given by Wilson. First we modify it suitably to take into account the fact that, in the presence of a sound wave, the distribution function for the electrons, $\bar{f}$, corresponding to local equilibrium is different from the equilibrium distribution function f_o in the absence of the sound wave. f_o is just the usual Fermi-Dirac distribution function. Secondly, in order to obtain the solution of the Boltzmann transport equation we expand f in a series of spherical harmonics,

$$f = f_o + \sum_{p,n} v^p C_p^n(v) Y_p^n(\theta,\phi) \frac{\partial f_o}{\partial E} u \,. \qquad (9)$$

We find that it is necessary to retain only the n=0 terms for incident plane dilational waves and the n=1 terms for incident plane shear waves. Using (9) in the transport equation results in an infinite set of coupled linear integral equations for the $C_p^n(v)$'s. Under certain simplifying assumptions valid at low temperatures, it is found that each

$C_p^n(v)$ relaxes towards its value corresponding to local equilibrium with a relaxation time, τ_p, such that

$$\tau_p \simeq \frac{2}{p(p+1)}\,\tau_1 , \qquad (10)$$

where τ_1 is the relaxation time for $C_1(v)$ and is just that appropriate for electrical conductivity. The set of coupled linear integral equations have now been reduced to a set of coupled linear algebraic equations.

The infinite set of coupled linear algebraic equations for the $C_p^n(v)$'s can be readily solved for certain special cases and the corresponding expressions for the attenuation derived. First, when $kv_o\tau_1 = k\ell_1 \ll 1$, one finds that the relaxation time appearing in the expressions for α is the same for both dilational and shear waves. It is found that this relaxation time is just $\tau_2 = \tau_1/3$. Secondly, for $k\ell_1 > 1$ but $\omega\tau_1 < 1$, the expressions for α are independent of the relaxation times. Thirdly, for $\omega\tau_1 \gg 1$, the expression for α of the dilational waves is just that of the previous case but the expression for the attenuation of the shear waves contains a relaxation time which is now to be interpreted as τ_1.

For the intermediate case, $k\ell_1 \sim 1$, the solutions of the coupled set of equations are much more difficult to obtain and we have not found a way to obtain analytical solutions. An attempt to obtain numerical solutions of these equations has not been made in this thesis.

In section 4 we discuss briefly the case when the collision integral arises from impurity scattering alone and find results which are qualitatively the same as those obtained in section 3 with respect

to relaxation times. In particular one can write $\tau_\rho' = a_\rho \tau_1'$, where a_ρ depends on the interaction potential between the electrons and the impurity ions. In the case of a simple screened coulomb potential, a_ρ depends only on the range of the interaction potential. The value of a_2 has been calculated as a function of the screening distance and is found to lie between one-third and unity.

Section 5 consists of a discussion of the results obtained in sections 2 to 4 in relation to experiment. The influence of electron-electron collisions is brought into this discussion on a qualitative level.

2. CALCULATION OF THE ATTENUATION AND THE DISPERSION RELATIONS ASSUMING THE EXISTENCE OF A SINGLE RELAXATION TIME

2.1 FORMULATION.

The first step in our calculations is to formulate the description of the electrons in a metal in terms of the Maxwell-Boltzmann transport equation.

Let $f_o(\vec{v})$ denote the Fermi-Dirac equilibrium distribution function for the electrons in the undisturbed metal, where

$$f_o(\vec{v}) = \left[\exp\left\{\left(\frac{mv^2}{2} - E_o\right)/k_oT\right\} + 1\right]^{-1}. \tag{11}$$

In (11), $E_o = \frac{mv_o^2}{2}$, m is the mass of the electron, v is its velocity, v_o is the velocity of an electron at the surface of the Fermi sphere, k_o is the Boltzmann constant, and T is the temperature. The number of free electrons, N_o, per unit volume and v_o are related by

$$N_o = \left(\frac{2m^3}{h^3}\right)\int f_o(\vec{v})d\vec{v} = \left(\frac{8\pi}{3}\right)\left(\frac{mv_o}{h}\right)^3. \tag{12}$$

It becomes convenient to take the number of free electrons to be equal to the number of metal atoms and this is done throughout.

Let an ultrasonic wave of wavelength, Λ, in the metal be characterized by a displacement $\vec{s}$ and a particle velocity $\vec{u}$. For a plane wave travelling in the x-direction.

$$\vec{s} = \vec{s}_o\exp\left[i(\omega t - kx)\right], \quad \vec{u} = \dot{\vec{s}}, \tag{13}$$

where $k = 2\pi/\Lambda$. The distribution function, $\bar{f}$, for the electrons in local equilibrium is given by

$$\bar{f} = \left[\exp\left\{\left(\frac{1}{2}m|\vec{v} - \vec{u}|^2 - E_o - \xi\right)/k_oT\right\} + 1\right]^{-1}. \tag{14}$$

That is, $\bar{f}$ is a Fermi sphere moving with the lattice and having a surface

energy $E_o + \xi_1$. The additional energy, ξ_1, is necessary to take into account the density flurtuations of the electrons caused by the passage of the ultrasonic wave through the metal. Here we assume that the effects of heat flow due to local changes in temperature of the electrons and of the lattice are negligible. We note that the attenuation per wavelength resulting from heat flow is at the most of the order $\left[C_p/C_v - 1\right]$, where C_p and C_v are the specific heats at constant pressure and volume, respectively. At liquid helium temperatures this factor is of the order of 10^{-8} or less. To first order in $\vec{u}$ and ξ_1, (14) may be written as

$$f = f_o - (m\vec{v}.\vec{u} + \xi_1) \frac{\partial f_o}{\partial E}, \qquad (15)$$

where $E = \frac{mv^2}{2}$.

Under the influence of the ultrasonic wave a certain electric field, $\vec{\mathcal{E}}$, will be set up and the stationary distribution function, f, for the electrons will be determined by the Maxwell-Boltzmann transport equation

$$\frac{\partial f}{\partial t} + \vec{v}.\vec{\nabla}_r f + \frac{e\vec{\mathcal{E}}}{m}.\vec{\nabla}_v f = \left[\frac{\partial f}{\partial t}\right]_{coll.}, \qquad (16)$$

where e is the electronic charge.*

For the first part we assume that f relaxes towards $\bar{f}$, (15), due to collisions with a relaxation time, τ,+ that is,

$$\left[\frac{\partial f}{\partial t}\right]_{coll} = -\frac{1}{\tau}.(f - \bar{f}). \qquad (17)$$

Let the solution of (16) be of the form

$$f = f_o - \vec{f}_1(\vec{v}).\vec{u}\frac{\partial f_o}{\partial E}, \qquad (18)$$

*,+ See next page for footnotes.

Substituting (17) and (18) into (16) and retaining only terms of first order in $\vec{\mathcal{E}}$, $\vec{u}$, and ξ_1 we have

$$\vec{f}_1(\vec{v}) \cdot \vec{u} = \frac{1}{(1 + i\omega\tau)} \cdot \frac{(e\tau\vec{v}\cdot\vec{\mathcal{E}} + m\vec{v}\cdot\vec{u}) + \xi_1}{(1 - ia'\cos\theta)}, \qquad (19)$$

where

$$a' = kv\tau(1 + i\omega\tau)^{-1}, \qquad (20)$$

and θ is the angle between $\vec{v}$ and the direction of propagation of the sound wave, (the x-axis.).

---

* In general, there will be also an induced magnetic field produced but in the first order theory in u developed here this will have a zero effect as is demonstrated as follows.
This term will be of the form.

$$\frac{e}{m}(\vec{v} \times \vec{H})\cdot\vec{\nabla}_v f$$

$$= \frac{e}{m}(\vec{v} \times \vec{H})\cdot\vec{\nabla}_v \left\{ f_o - \vec{f}_1(\vec{v})\cdot\vec{u}\frac{\partial f_o}{\partial E} \right\}$$

$$= \frac{e}{m}(\vec{v} \times \vec{H})\cdot\vec{\nabla}_v f_o + \text{higher order terms in u.}$$

$$= \frac{e}{m}(\vec{v} \times \vec{H})\cdot m\vec{v}\frac{\partial f_o}{\partial E} = 0,$$

since $(\vec{v} \times \vec{H})\vec{v} = 0$ and since $\vec{H}$ will be proportional to $\vec{u}$. Hence the induced magnetic field may be neglected throughout.

+ Holstein, (1959) [12], has given a theoretical justification of the form of $\bar{f}$ and his treatment will be discussed more fully in section 3.1.

The electronic current density, $\vec{J}_e$, and the number of electrons, N, per unit volume at a point in space are given by;

$$N = \left(\frac{2m^3}{h^3}\right) \int f(\vec{v})d\vec{v} \quad \text{and} \quad \vec{J}_e = \left(\frac{2em^3}{h^3}\right)\int \vec{v}f(\vec{v})d\vec{v} \,. \qquad (21)$$

The equation of continuity demands that

$$\frac{\partial}{\partial t}(N_e) + \operatorname{div} \vec{J}_e = 0 \,, \qquad (22a)$$

On integrating (14) ~~throughout~~ over all velocities of the electron, we see that (22a) is equivalent to

$$\int (f - \bar{f})\, d\vec{v} = 0 \,, \qquad (22b)$$

The electric field, $\vec{\mathcal{E}}$, is related to the total current density, $\vec{J} = \vec{J}_e - N_o e\vec{u}$, by Maxwell's equations (in Gaussian units),

$$\begin{aligned} \text{curl curl } \vec{\mathcal{E}} &= \text{grad div } \vec{\mathcal{E}} - \nabla^2 \vec{\mathcal{E}} \\ &= -\frac{1}{c^2}\left[4\pi(\dot{\vec{J}}_e - N_o e\dot{\vec{u}}) + \ddot{\vec{\mathcal{E}}}\right]. \end{aligned} \qquad (23)$$

Equations (22) and (23) used in conjunction with (19) and (21) are sufficient to determine $\vec{f}_1$, ξ_1, and $\vec{\mathcal{E}}$, completely, in terms of the particle velocity $\vec{u}$ associated with the ultrasonic wave.

Next, we set up the equations of motion of the ions. The equations of motion for wave propagation in the x-direction in an isotropic solid may be written as

$$\rho\ddot{\vec{s}} = \rho\beta\frac{\partial^2 s}{\partial x^2} - N_o e\vec{\mathcal{E}} - \vec{I}_c \,, \qquad (24)$$

where $\rho (= N_o M)$ is the ionic density of the metal, -e is the ionic charge, and β is the square of the velocity of the wave in the absence of the electron-ion interaction, that is, for $\vec{\mathcal{E}}$ and $\vec{I}_c$ zero in (24). The second term on the right hand side of (24) is due to the electric field and is the force on a unit volume, the electric force. The third term on the right hand side of (24) is the impulsive force resulting from the direct collisions of the ions with the electrons.

The term $\vec{I}_c$ can be determined as follows. Due to the collisions between the electrons and the ions, the momentum of the electrons per unit volume changes with time at a rate

$$\vec{I}_c = \left(\frac{2m^3}{h^3}\right) \int_{\text{coll.}} m\vec{v} \left[\frac{\partial f}{\partial t}\right] d\vec{v} \quad .$$

Substituting (17) and (21) into the above we have

$$\vec{I}_c = -\left(\frac{2m^*}{h^3 \tau}\right) \int \vec{v}(f - \bar{f}) \, d\vec{v}$$

$$= \frac{1}{\tau} \left[-\frac{m\vec{J}_e}{e} + N_o m\vec{u} \right] . \qquad (25)$$

The momentum imparted to the ions per unit time per unit volume by the electrons in collisions with the ions is just $-\vec{I}_c$.

On substituting (13) into (24) and putting $k = k_1 - ik_2$ one can obtain the velocity, $V = \omega/k_1$, and the attenuation, $\alpha = 2k_2$, in nepers/cm. When $k_2 << k_1$, as is the case in the problem considered here, we rewrite (24) in the form;

$$\omega^2 = \beta k^2 + Z$$

$$= \beta(k_1^2 - k_2^2 - 2ik_1k_2) + Z_1 + iZ_2 , \qquad (26)$$

where

$$Z = \frac{(N_o e \vec{\mathcal{E}} + \vec{I}_c) \cdot \vec{s}}{\rho s^2} \quad . \tag{27}$$

Equating real and imaginary parts in (26) then

$$\alpha = \frac{1}{\beta k_1} Z_2 = \frac{V}{\beta \omega} Z_2 \tag{28}$$

and

$$V^2 = \frac{\omega^2}{k_1^2} = \beta \left[1 - \frac{k_2^2}{k_1^2} \right] + \frac{Z_1}{k_1^2} \quad . \tag{29}$$

2.2 INCIDENT PLANE DILATIONAL WAVE.

We represent a dilational wave propagating in the x-direction by (13) where now $\vec{s}_o = s_o \underline{i}$. Both $\vec{u}$ and $\vec{\mathcal{E}}$ will be along the x-direction and curl$\vec{\mathcal{E}}$ will be zero. In this case, (19) becomes

$$f_1(\vec{v})\, u = \frac{(e\tau v_x \mathcal{E} + m v_x u)}{(1 + i\omega\tau)(1 - ia'\cos\theta)} + \xi_1', \tag{30}$$

Setting $\frac{\partial f_o}{\partial E} = -\delta(E - E_o)$, (21) becomes

$$J_e = \frac{3N_o e g}{(1 + i\omega\tau)\, a^2 m v_o}\left[v_o(e\tau\mathcal{E} + mu) + ia\xi_1\right], \tag{31}$$

and

$$N = N_o + i\frac{3N_o g}{2(1 + i\omega\tau)aE_o}\left[v_o(e\tau\mathcal{E} + mu) + ia\xi_1\left(1 - \frac{1}{g}\right)\right], \tag{32}$$

where

$$g = 1 - \frac{1}{2}\int_{-1}^{1}\frac{dx}{1 - iax} \quad \text{and} \quad a = \frac{v_o k \tau}{1 + i\omega\tau}, \tag{33}$$

and $x = \cos\theta$.

Using (22a) we may now solve for ξ_1' and obtain

$$\xi_1' = i\frac{v_o g (e\tau\mathcal{E} + mu)}{a(g + i\omega\tau)}. \tag{34}$$

Using (31) and (34) in (23) we obtain for $\mathcal{E}$,

$$e\mathcal{E} = \frac{mu}{\tau}\frac{(\chi - 1)}{(1 - i\gamma\chi\omega\tau)}, \tag{35}$$

where $\gamma = \frac{m}{4\pi N_o e^2}$,

and

$$\chi = \frac{(1 + i\omega\tau)\, a^2}{3g}\left(1 + \frac{g}{i\omega\tau}\right) = \frac{kla}{3g}\left(1 + \frac{g}{i\omega\tau}\right), \quad (36)$$

Substituting (31), (34), and (35) into (25), then

$$I_c = \frac{N_o m u}{\tau^2}\,\frac{i\gamma\omega(\chi - 1)}{(1 + i\gamma\omega\chi/\tau)}\,. \quad (37)$$

Hence, from (27)

$$Z = \frac{(N_o e \mathcal{E} + I_c)s}{\rho s^2}$$

$$= \frac{i\omega m N_o}{\rho\tau}\,\frac{(\chi - 1)(1 + i\gamma\omega/\tau)}{(1 + i\gamma\omega\chi/\tau)}\,, \quad (38)$$

First, we note that γ appearing above is the reciprocal of the square of the plasma frequency ω_p and so $\gamma \simeq 10^{-32}$ in metals. At the ultrasonic frequencies currently available and for the values of the relaxation time, τ, that occur in metals, the terms $(\gamma\chi\omega/\tau)$ and $(\gamma\omega/\tau)$ may be neglected. This is demonstrated in the following.

Set

$$\chi = \chi_1 + i\chi_2\,, \quad (39)$$

Then (38) may be written,

$$Z = \frac{i\omega m N_o}{\rho\tau}\left[\left\{(\chi_1 - 1) + \frac{\gamma^2\omega^2}{\tau^2}(\chi_1^2 + \chi_2^2 - \chi_1)\right\} + i\left\{\chi_2\left(1 + \frac{\gamma^2\omega^2}{\tau^2} - \frac{\gamma\omega}{\tau}\right)\left((\chi_1 - 1)^2 + \chi_2^2\right)\right\}\right] \times \left[1 - 2\frac{\gamma\omega}{\tau}\chi_2 + \frac{\gamma^2\omega^2}{\tau^2}(\chi_1^2 + \chi_2^2)\right]^{-1}. \quad (40)$$

From (33),

$$g = 1 - \frac{1}{2}\int_{-1}^{1} \frac{dx}{1-iax} = 1 + \frac{1}{2ia}\ln\left(\frac{1-ia}{1-ia}\right) = 1 - \frac{1}{a}\tan^{-1}a, \tag{41}$$

and therefore,

$$\chi = \frac{k\ell a}{3}\left[\frac{a}{a - \tan^{-1}a} + \frac{1}{i\omega\tau}\right]. \tag{42}$$

For $a << 1$, $k, \ell << 1$, and $\omega\tau << 1$,

$$\tan^{-1} a = a - a^3/3 + a^5/5 - \ldots \tag{43}$$

Hence,

$$\chi = (1 + i\omega\tau)\left[(1 + \frac{3a^2}{5} + \ldots) + \frac{a^2}{3i\omega\tau}\right]$$

$$= \left[1 + \frac{4k_1^2\ell^2}{15}\left\{\left(1 - \frac{k_2^2}{k_1^2}\right) - \frac{5k_2}{2\omega\tau k_1} - \frac{2\omega\tau k_2}{k_1} - \omega^2\tau^2 + \ldots\right]\right.$$

$$+ i\left[\omega\tau - \frac{k_1^2\ell^2}{15}\left\{\frac{5}{\omega\tau}\left(1 - \frac{k_2^2}{k_1^2}\right) + 4\left(\omega\tau + \frac{2k_2}{k_1}\right) + \ldots\right]\right]. \tag{44}$$

Now for metals at liquid helium temperatures $\tau > 10^{-12}$ second and $\gamma \sim 10^{-32}$. The ultrasonic frequencies available are less than 10^{11} radians per second and hence $\frac{\gamma\omega}{\tau} < 10^{-9}$. Thus from (44), $\chi_1 \sim 1$ and $\chi_2 \sim \omega\tau << 1$, all the terms containing factors of $\frac{\gamma\omega}{\tau}$ may be neglected.

In this case, $k_1\ell << 1$, using (28), (41), (40), and (44) we have.

$$2k_2 = \frac{m\omega}{\beta_\ell k_1 M\tau}(\chi_1 - 1) = \frac{m\omega}{\beta_\ell k_1 M\tau}\frac{4k_1^2\ell^2}{15}\left[1 - \frac{5k_2}{2\omega\tau k_1} - \ldots\right].$$

Solving for $\alpha_\ell = 2k_2$ yields,

$$\alpha_\ell = \frac{4mv_o^2 k_1\omega\tau}{15M}\left(\beta_\ell + \frac{mv_o^2}{3M}\right)^{-1} = \frac{4mv_o^2\omega^2\tau}{15MV_\ell^3}, \tag{45}$$

since $k_1 = \frac{\omega}{V_\ell}$ and $V_\ell^2 \simeq \beta_\ell + \frac{mv_o^2}{3M}$ as is seen immediately.

Again neglecting the terms containing χ/τ, we have

$$\frac{\omega^2}{k_1^2} = \beta_\ell\left(1 - \frac{k_2^2}{k_1^2}\right) - \frac{m\omega}{Mvk_1^2}\chi_2$$

$$= \beta_\ell\left(1 - \frac{k_2^2}{k_1^2}\right) - \frac{m\omega^2}{Mk_1^2} + \frac{mv_o^2}{3M}\left(1 - \frac{k_2^2}{k_1^2}\right) + \frac{4mv_o^2}{15M}\omega\tau\left(\omega\tau + \frac{2k_2}{k_1}\right) + \cdots$$

Therefore,

$$\frac{\omega^2}{k_1^2}\left(1 + \frac{m}{M}\right) = \left[\left(\beta_\ell + \frac{mv_o^2}{3M}\right) + \frac{4mv_o^2}{15M}\left(\omega^2\tau^2 + \omega\tau\frac{2k_2}{k_1}\right) + \cdots\right] \times \left(1 - \frac{k_2^2}{k_1^2}\right)$$

and

$$V_\ell^2 = \frac{\beta_\ell(1 - \alpha_\ell^2/4k_1^2)}{(1 + m/M)}\left[1 + \frac{mv_o^2}{3M\beta_\ell} + \frac{4mv_o^2}{15M}\left(\omega^2\tau^2 + \omega\tau\frac{\alpha_\ell}{k_1}\right) + \cdots\right]$$

$$= \left(\beta_\ell + \frac{mv_o^2}{3M}\right)\frac{(1 - \alpha_\ell^2/4k_1^2)}{(1 + m/M)}\left[1 + \frac{\alpha_\ell}{k_1}(\omega\tau + \alpha_\ell/k_1) + \cdots\right]. \qquad (46)$$

Similarly, for $k_1\ell >> 1$ and arbitrary $\omega\tau$, we expand $\tan^{-1} a$ to obtain

$$\tan^{-1} a = \frac{\pi}{2} - \frac{1}{a} + \frac{1}{3a^3} + \cdots$$

Therefore,

$$\chi = \left[-\frac{2k_1k_2\ell^2}{3\omega\tau} + \frac{\pi k_1\ell}{6} + \frac{1}{3}\left(\frac{\pi^2}{4} - 1\right) + \cdots\right]$$

$$+ \quad i\left[-\frac{k_1^2\ell^2}{3\omega\tau}\left(1-\frac{k_2^2}{k_1^2}\right)+\frac{\pi k_2\ell}{6}+\frac{\omega\tau}{3}\left(\frac{\pi^2}{4}-1\right)+\cdots\right]. \qquad (47)$$

Hence $\chi_1 \sim k_1\ell$ and $\chi_2 \sim k_1\ell \times 10^3$ and thus we may again neglect the terms containing $\frac{\delta\omega}{\tau}$ in (40) for practical frequencies ($\omega < 10^{11}$ radians per second).

Therefore, (28), (40), and (47) give

$$\alpha_\ell = \frac{\pi m v_o \omega}{6M}\left(\beta_\ell + \frac{m v_o^2}{3M}\right)^{-1} = \frac{\pi m v_o \omega}{6M V_\ell^2}, \qquad (48)$$

and

$$V_\ell^2 = \frac{(\beta_\ell + m v_o^2/3M)}{\left[1+\frac{m}{M}\left(\frac{\pi^2}{4}-1\right)\right]}\left[1-\frac{\alpha_\ell^2}{2k_1^2}+\cdots\right]\left[1-\frac{\alpha_\ell^2}{4k_1^2}\right]. \qquad (49)$$

Finally, from the above we see that we may set

$$\alpha_\ell = \frac{m}{MV_\ell\tau}\left[\frac{k_1^2\ell^2}{3}\left(\frac{1}{g_o}-1\right)-1\right] \qquad (50)$$

over the whole frequency range and where

$$g = 1 - (k_1\ell)^{-1}\tan^{-1}(k_1\ell).$$

We note that (50) is precisely the result obtained by Pippard, (1955) [6], and Steinberg, (1958a) [7].

2.3. INCIDENT PLANE SHEAR WAVES.

The development is just the same as that used in the last section for incident plane dilational waves. We consider a plane shear wave travelling in the x-direction and with a displacement, $\vec{s}$, and particle velocity, $\vec{u}$, associated with the wave in the y-direction. In this case there are no density changes and hence no electric fields resulting from a space charge. However, the total current density, $\vec{J} = \vec{J}_e - N_o e\vec{u}$, will, in general, be different from zero. In this case, magnetic fields will be generated and from these electric fields by induction. The effect of the magnetic fields can be neglected both in the Boltzmann transport equation and in the equation determining the ion motion, as we have already demonstrated. Hence we have only to determine $\vec{\mathcal{E}}$. $\vec{J}$ and $\vec{\mathcal{E}}$ will both be parallel to $\vec{u}$ or to the y-axis. Further, div.$\vec{\mathcal{E}} = 0$ and the equation of continuity. (21a), requires that $\zeta_1 = 0$, (no space charge).

From (19) we have

$$f_1(\vec{v})u = \frac{(e\tau\mathcal{E} + mu)\ v_y}{(1 + i\omega\tau)\ (1 - ia'\cos\theta)}, \tag{51}$$

where $v_y = v\sin\theta\cos\phi$ and everything else has been defined in section 2.2. From (20),

$$J_e = N_o e \left[\frac{e\tau\mathcal{E}}{m} + u \right] b , \tag{52}$$

where

$$b = \frac{3}{2(1 + i\omega\tau)} \left[1 - \varepsilon - \frac{\varepsilon}{a^2} \right] . \tag{53}$$

Hence,

$$e\mathcal{E} = \frac{mu}{\tau} k_o^2 (1 - b) \left[k_o^2 b + k^2 - \omega^2/c^2 \right]^{-1}, \tag{54}$$

where here, $k_o^2 = \frac{4\pi i \omega \sigma}{c^2}$ and $\sigma = \frac{N_o e^2 \tau}{m}$. (55)

ince $c >> V_s$, we may neglect ω^2/c^2 compared to k^2 throughout the rest of this section.

Continuing,

$$I_c = \frac{N_o mu}{\tau} (1 - b) k^2 \left[k_o^2 b + k^2 \right]^{-1}. \tag{56}$$

Thus the equation of motion for the ions, (24) becomes

$$\rho \ddot{s} = \rho \beta_s \frac{\partial^2 s}{\partial x^2} - \frac{N_o mu}{\tau} (1 - b) \frac{(k_o^2 + k^2)}{(k_o^2 b + k^2)}. \tag{57}$$

In (57) β_s is the square of the velocity of the shear waves in the absence of the electron-ion interaction and k_o is the reciprocal of the classical skin depth of electromagnetic waves of angular frequency ω.

The attenuation, α_s, and the velocity, V_s, of the shear waves depend upon the relative magnitude of k and k_o. We consider first the case in which $|k^2/k_o^2|$ and $|k^2/k_o^2 b|$ are both much less than unity. Now (57) becomes

$$\rho \ddot{s} = \rho \beta_s \frac{\partial^2 s}{\partial x^2} - \frac{i\omega N_o ms}{\tau} \left(\frac{1}{b} - 1 \right), \tag{58}$$

Setting $b = b_o - ib_1$, then

$$2k_2 = \frac{\omega m}{\beta_s M k_1 \tau} \left[\frac{b_o}{b_o^2 + b_1^2} - 1 \right], \tag{59}$$

and

$$\omega^2 = \beta_s(k_1^2 - k_2^2) + \frac{\omega m}{M\tau}\left[\frac{b_1}{b_o^2 + b_1^2}\right]. \tag{60}$$

There are now two cases of interest in this region. These are, first, $k\ell \ll 1$ and, second, $k\ell \sim 1$.

For $k\ell \ll 1$,

$$b = \frac{1}{(1 + i\omega\tau)}\left[1 - \frac{a^2}{5} + \frac{3a^4}{35} + \ldots\right]. \tag{61}$$

Thus

$$\alpha_s = 2k_2 = \frac{\omega m k_1 \ell^2}{5\beta_s M\tau} = \frac{m v_o^2 \omega^2 \tau}{5MV_s^3}, \tag{62}$$

and

$$\omega^2 = \beta_s(k_1^2 - k_2^2) - \frac{\omega m}{M\tau}\left[\omega\tau - \frac{k_1^2\ell^2}{5}\left(\omega\tau + \omega\tau\frac{k_2^2}{k_1^2} + \frac{2k_2}{k_1} - \omega^3\tau^3 + 2\omega^2\tau^2\frac{k_2}{k_1}\right) - \frac{8k_1^4\ell^4}{175} + \ldots\right],$$

Therefore,

$$V_s^2 = \frac{\omega^2}{k_1^2} = \frac{\beta_s(1 - k_2^2/k_1^2)}{(1 + m/M)}\left[1 + \frac{m v_o^2 \omega^2 \tau^2}{5M} + \frac{\alpha^2}{k_1^2} + \ldots\right]$$

$$= \frac{\beta_s(1 - \alpha^2/4k_1^2)}{(1 + m/M)}\left[1 + \frac{m v^2 \omega^2 \tau^2}{5M} + \frac{\alpha^2}{k_1^2} + \ldots\right]. \tag{63}$$

For $k_1\ell \sim 1$, then

$$b = \frac{3}{2(1 + i\omega\tau)}\left[\frac{\pi}{2a} - \frac{2}{a^2} + \frac{1}{a^2}\,\frac{1}{3} + \frac{\pi}{2} + \ldots\right]. \tag{64}$$

Hence,

$$\alpha_s = \frac{\omega m}{M\tau\beta_s k_1}\left[\frac{4k_1\ell}{3\pi} + \cdots\right] = \frac{4mv_o\omega}{3\pi MV_s^2}, \tag{65}$$

and

$$V_s^2 = \frac{\omega^2}{k_1^2} = \frac{\beta_s(1 - k_2^2/k_1^2)}{(1 + 16m/3\pi^2M)}\left[1 + \frac{m}{M}\left\{\frac{4v_o k_2\ell}{3\pi V_s} - 2\left(1 + \frac{2}{3\pi} - \frac{16}{\pi^2}\right)\right\} + \frac{mv_o}{MV_s}\frac{k_2\ell}{k_1^3\ell^3}(\omega^2\tau^2 - 1)\left(1 + \frac{2}{3\pi} - \frac{16}{\pi^2}\right) + \cdots\right], \tag{66}$$

As $k\ell$ increases further, it becomes invalid to neglect k^2 compared to $|bk_o^2|$. The critical frequency, ω_c, at which $k^2 = |kb_o^2|$ is given by

$$\omega_c = \left(\frac{3\pi^2 V_s^3 \sigma}{\ell c^2}\right)^{1/2}, \tag{67}$$

To obtain this value we have used (64) and have set $b_o = \frac{3\pi}{4k\ell}$.

For most metals $\frac{\omega_c}{2\pi} \sim 10^9$ cycles per second. At frequencies close to ω_c the attenuation starts to diminish with increase of frequency until eventually, when $k^2 >> k_o^2$ and $\omega\tau >> 1$, it becomes independent of frequency, α_s and V_s are now given by

$$\alpha_s = \frac{m}{MV_s\tau}, \tag{68}$$

and

$$V_s^2 = \beta_s\left[1 + \frac{3mV_s^2}{Mv_o^2}\frac{1}{\omega^2\tau^2} + \cdots\right]. \tag{69}$$

It will be noticed that at low frequencies, $k\ell < 1$, $k\ell \sim 1$, and $\omega\tau \ll 1$, the attenuation and dispersion of shear waves are due to the induced electric force. Whereas, at very high frequencies, $\omega\tau \gg 1$, they are due solely to the collision term, I_c, that is, they are due to the transfer of momentum to the ions from the electrons by direct collisions. This is in contrast to the case of dilational waves for which the attenuation and dispersion arise from the induced electric fields at all frequencies.

2.4 DISCUSSION.

The expressions for the attenuation, (45) and (48) for dilational waves and (62), (65) and (68) for shear waves, are just the same as those obtained by Pippard, (1955) [6], and Steinberg, (1958a) [7]. In Pippard's case, where he considers the relaxation time for the electron velocity, it seems reasonable to identify such a relaxation time with τ_1, that appropriate for electrical conductivity. However, as pointed out by Mason and Bömmel, (1956) [4], if this is done and if the theoretical attenuation curve for tin is fitted at low frequencies, $k\ell_1 \ll 1$ and $\omega\tau_1 \ll 1$, then the theoretical curve is small by a factor of about 2.5 to 7.5 for dilational waves in the region $k\ell_1 > 1$ and it is small by a factor 1.3 to 1.7 for shear waves in the region $k\ell_1 > 1$, $\omega\tau_1 < 1$.

Steinberg has given an explanation for this discrepancy by interpreting the relaxation time appropriate for the distribution function and therefore that appearing in his expressions for attenuation to be $\tau_1/2$. If we take expressions (45), (48), (62), (65), (68) and set $\tau = \tau_1/2$ and follow Mason and Bömmel's procedure of fitting the theoretical curves to the experimental curves, then the attenuation will be just twice as large as that given from Pippard's formulation in the higher frequency range, $k\ell_1 > 1$ and $\omega\tau_1 < 1$, and four times as large for shear waves at extremely high frequencies, $k\ell_1 >> 1$, $\omega\tau_1 >> 1$. However, Steinberg's argument is erroneous as can be seen readily by verifying that, once one has characterized the collision integral by a single relaxation time, as in (17) f, v_0' and E_0' all relax toward their equilibrium values at the same rate.

We shall show in the following sections that the explanation of the experimental results lies in the fact that the collision integral cannot be characterized by a single relaxation time for the whole frequency range. In particular, it turns out for the low frequency region, $k\ell_1 \ll 1$, $\omega\tau_1 \ll 1$, that a relaxation time, τ_2, appears in the expressions for the attenuation and we may write $\tau_2 \simeq a\tau_1$, where a lies between 1/3 and 1. The precise value of a depends upon the details of the scattering mechanisms.

Before we take up the calculations of the relaxation times, we mention that our results for the dependence of the velocities of the sound waves on the frequency are qualitatively the same as the results obtained by Steinberg. Since the velocity changes over the entire frequency range are of the order of the mass of the electron over the mass of the ion, that is, of the order of 10^{-5}, we shall not discuss them further. We should note, however, in comparing (46) and (49) with the corresponding expressions in section 4 of Steinberg's paper, [7], one must identify his c_s^2 with our $\beta_\ell + \frac{1}{3}\left(\frac{mv_o^2}{M}\right)$, since this is the effective elastic constant seen by the dilational sound wave. It is interesting to note that, in the case of the dilational waves, the electron-ion interaction contributes a term

$$v_o\left(\frac{m}{3M}\right)^{1/2} = \rho^{-1/2}\chi \text{ (the bulk modulus of the electron gas)}^{1/2}$$

to the velocity independently of the relaxation processes due to collisions. This term has its origin in the fact that when a dilational wave passes through a metal both the ionic and electronic densities

vary from point to point. Since the electrons obey the Pauli exclusion principle, their contribution to the bulk modulus becomes appreciable. In the case of shear waves, no density variations occur and hence no equivalent term is obtained.

If we do not neglect the term $(\gamma\chi_{\omega}/\tau)$ appearing in the denominator of (39) but do neglect the terms which give rise to the attenuation, we obtain

$$\frac{V_t^2}{} = \beta_t + \frac{mv_o^2}{3M} \left(1 + \gamma v_o^2 k^2/3 \right)^{-1}.$$

This expression is approximately valid at all frequencies as long as the metal can be described by the Sommerfeld free electron model. The above expression has been derived by Bhatia, (1955) [10], in a much simpler manner in another connection.

3. CALCULATION OF THE RELAXATION TIMES FOR THE ELECTRON-PHONON INTERACTION.

3.1 FORMULATION.

We again formulate the problem in terms of the Maxwell-Boltzmann transport equation and employ the same physical concepts as used in section 2. Instead of making the assumption that the collision integral is characterized by a single relaxation time, we set up the collision integral so that it takes into account the collisions between the electrons and the lattice vibrations. To do this we start with equation 9.33.6 in Wilson's "Theory of Metals", (1954) [11], and proceed from there. Using the assumption that the equilibrium distribution is given by (13) we set the collision integral zero for this function. The experimental data also suggests that there is very little energy dissipation due to thermal currents and hence we use the phonon distribution which corresponds to thermal equilibrium in the undisturbed state of the metal. (See page 21) Equation 9.33.6 from Wilson reads

$$\left[\frac{\partial f}{\partial t}\right]_{coll} = \frac{C^2}{(2G+1)^3 \, Mh} \sum_{\vec{q}} \left| \sum_{j} \frac{\vec{q}\cdot\vec{e}_{\vec{q}j}}{\nu_{\vec{q}j}^{1/2}} \right|^2 \Big[\Big\{ (N_{\vec{q}}+1) f(\vec{K}+\vec{q})(1-f(\vec{K})) - N_{\vec{q}} f(\vec{K})(1-f(\vec{K}+\vec{q})) \Big\} \Omega(E_{\vec{K}} - E_{\vec{K}+\vec{q}} + h\nu_{\vec{q}}) + \Big\{ N_{\vec{q}} f(\vec{K}-\vec{q})(1-f(\vec{K})) - (N_{\vec{q}}+1) f(\vec{K})(1-f(\vec{K}-\vec{q})) \Big\} \Omega(E_{\vec{K}} - E_{\vec{K}-\vec{q}} - h\nu_{\vec{q}}) \Big], \tag{70}$$

where the $\vec{e}_{\vec{q}j}$ are unit vectors defining the direction of vibration for each of the three acoustical modes designated by j. The 3(s-1) optical modes are absent in a metal of all the same type atom. $\vec{q}$ is the phonon wave vector, M in the mass of the ion, h is Planck's constant, $\nu_{\vec{q}j}$ is the frequency of this lattice vibration, $(2G+1)^3$

is the number of atoms in a cyclical three dimensional crystal forming a Bravais lattice, and C^2 is a constant measuring the strength of the electron-ion interaction and is of the order of a few electron volts. The summation goes over all possible values of $\vec{q}$ to take into account both the emission and the absorption of thermal phonons by the electrons. The $N_{\vec{q}}$ represent the probability that there is a phonon designated by $\vec{q}$ at a given temperature, T. For thermal equilibrium

$$N_{\vec{q}} = \left[\exp(h\nu_{\vec{q}}/k_o T) - 1\right]^{-1}, \tag{71}$$

where k_o is the Boltzmann constant and

$$\Omega(x) = \frac{\sin(xt/\hbar)}{x/\hbar}, \tag{72}$$

and hence acts as a delta function conserving energy. $f(\vec{K})$ is the electron distribution function and represents the probability that the electron state designated by $\vec{K}$ is occupied. We note here that there is a two-fold spin degeneracy which must be taken into account but will not appear explicitly in the Boltzmann equation.

If we now set

$$f = \bar{f} - f_2 u \frac{\partial f_o}{\partial E}, \tag{73}$$

and set the collision integral equal to zero for $\bar{f}$, then (70) reduces to 9.33.7 in Wilson, that is,

$$\left[\frac{\partial f}{\partial t}\right]_{coll} = \frac{C^2 \Delta}{8\pi^3 M h k_o T} \int \frac{|\vec{q}|^2}{\nu_{\vec{q}}} N_{\vec{q}} \left[f_o(\vec{K}) \left\{ 1 - f_o(\vec{K}+\vec{q}) \right\}
$$

$$x\,\Omega(E_{\vec{K}}-E_{\vec{K}+\vec{q}}+h\nu_{\vec{q}}) + f_o(\vec{K}+\vec{q})\left\{1 - f_o(\vec{K})\right\}\Omega(E_{\vec{K}}-E_{\vec{K}+\vec{q}}-h\nu_{\vec{q}})\Big]$$

$$x\left[f_2(\vec{K}+\vec{q})u\frac{\partial f_o(E_{\vec{K}+\vec{q}})}{\partial E_{\vec{K}+\vec{q}}} - f_2(\vec{K})u\frac{\partial f_o}{\partial E}\right] d\vec{q}\,, \qquad (74)$$

where we now use f_o rather than $\bar{f}$, since we are interested only in a first order theory in u, and Δ is the volume of a unit cell.

In working further with this integral we shall assume that $E_{\vec{K}}$ is a quadratic function of $|\vec{K}|$ and has spherical symmetry, and that $\nu_{\vec{q}}$ is a function of $|\vec{q}|$ alone with the values of $|\vec{q}|$ lying inside a sphere of radius q_o. For an elastic continuum,

$$q_o = \frac{(6\pi^2)^{1/3}}{a}\,, \quad q = \frac{q_o}{\nu_D}\nu_{\vec{q}}\,, \text{ and } h\nu_D = k_o\Theta\,, \qquad (75)$$

where ν_D is the maximum frequency supported by the lattice, Θ is the Debye temperature for the solid, and a is the linear dimension of a unit cell in the solid.

We now write $\vec{q}$ in terms of (q,γ,ω) where q is the length of $\vec{q}$, γ is the angle between $\vec{q}$ and $\vec{K}$, and ω represents an azimuthal angle about $\vec{K}$ taken as the polar axis. With this in mind we can write

$$K_x = K\cos\theta\,,$$

$$K_y = K\sin\theta\cos\phi\,,$$

$$K_z = K\sin\theta\sin\phi\,,$$

and

$$q_x = q(\cos\theta\cos\gamma + \sin\theta\sin\gamma\cos\omega)\,,$$

$$q_y = q(\sin\theta\cos\phi\cos\gamma - \cos\theta\cos\phi\sin\gamma\cos\omega + \sin\phi\sin\gamma\sin\omega)\,,$$

$$q_z = q(\sin\theta\sin\phi\cos\gamma - \cos\theta\sin\phi\sin\gamma\cos\omega - \cos\phi\sin\gamma\sin\omega)\,.$$

Since the Ω's act as delta functions in energy, then we can

carry out the integration over γ and we write

$$\cos\gamma_+ = (h\nu_q - E_q)/\hbar vq ,$$

and

$$\cos\gamma_- = (h\nu_q + E_q)/\hbar vq , \tag{76}$$

where the plus sign on γ indicates the absorption of a phonon by the electron and the minus sign the emission of a phonon. $E_q = \frac{\hbar^2 q^2}{2m}$ is the energy of an electron with the wave vector $\vec{q}$.

As demonstrated by Wilson, we must set

$$\Omega(E_{\vec{K}} - E_{\vec{K}-\vec{q}} \pm h\nu_q) \, d\cos\gamma = \Omega\left\{q\frac{dE}{dK}\cos\gamma + E_q \pm h\nu_q\right\} d\cos\gamma$$

$$= \frac{h}{2\pi q\left(\frac{dE}{dK}\right)} \frac{\sin y}{y} \, dy = \frac{h}{2q\left(\frac{dE}{dK}\right)} \delta(y) \, dy , \tag{77}$$

where

$$y = \frac{2\pi t}{h}\left(E_q \pm h\nu_q + q\frac{dE}{dK}\cos\gamma\right)$$

and

$$E_K = \frac{\hbar^2 K^2}{2m} .$$

In this event,

$$\left[\frac{\partial f}{\partial t}\right]_{coll} = \frac{C^2\Delta}{8\pi^2 Mhv}\int \frac{q^3 dq d\omega}{\nu_q}\left[(N_q + 1)\frac{(e^{\eta}+1)}{(e^{\eta+z}+1)}\left\{f_2(\vec{K}) - f_2(\vec{K}+\vec{q}_+)\right\}u\frac{\partial f_o}{\partial E} + N_q\frac{(e^{\eta}+1)}{(e^{\eta-z}+1)}\left\{f_2(\vec{K}) - f_2(\vec{K}-\vec{q}_-)\right\}u\frac{\partial f_o}{\partial E}\right] , \tag{78}$$

where the plus sign indicates that $\cos\gamma_+$ must be used and the minus sign that $\cos\gamma_-$ must be used. Also we have set

$$\eta = \frac{(E - E_o)}{k_o T} \quad \text{and} \quad z = \frac{h\nu_q}{k_o T} . \tag{79}$$

Restricting the values of q as in (75) we can reduce '78) to

$$\left[\frac{\partial f}{\partial t}\right]_{coll} = \frac{u\partial f_o}{\partial x}\frac{1}{\Lambda K^2 v}\left(\frac{T}{\Theta}\right)^3\int_{\Theta/T}^{\Theta/T}\frac{z^2dz}{|1-e^{-z}|}\frac{(e^{\eta}+1)}{(e^{\eta+z}+1)}\left\{f_2(\vec{K})\right.$$

$$\left. - \left\langle f_2\,(\vec{K}+\vec{q}_+)\right\rangle_\omega\right\}, \qquad (80)$$

where

$$\frac{1}{\Lambda} = \left(\frac{3}{4\pi}\right)^{1/3}\frac{3h^2c^2}{4\pi ak_o\Theta},$$

and the last term indicates the average value over ω. In setting the limits of integration we have used the fact that for metals $N > N_o/4$ and for this case the restriction on ν is $h\nu \leq k_o\Theta$. However, for semi-metals and semi-conductors $N < N_o/4$ and the more restrictive condition $h\nu \leq hv_oK/\pi$ must be used. This results in the limit Θ/T being replaced by $\left(\frac{2E}{D}\right)^{1/2}\frac{\Theta}{T}$ and this generalization can be readily carried out in the subsequent work, although, it has been omitted here. We are now in a position to start our solution.

[As we have mentioned, Holstein, (1959) [12], has very recently proven that $\bar{f}$, (14), is actually the distribution function corresponding to local equilibrium, towards which the electrons relax in the presence of the ultrasonic wave, through the electron-phonon interaction. Before proceeding further we show that Holstein's collision integral reduces to (80) above on which our considerations are based.

He finds that *one must set* the lattice displacement vector $\vec{R}_{\vec{g}}$ equal to the sum of the displacement due to the thermal motion, $\vec{R}_{\vec{g},th}$, and the displacement due to the ultrasonic wave, $\vec{s}$, that is, $\vec{R}_{\vec{g}} = \vec{R}_{\vec{g},th} + \vec{s}$. He then considers the change in potential energy

due to the displacement of the ions by expanding the potential energy in a power series in $\vec{r}_{\vec{q},th}$ and $\vec{s}$ and retaining all terms linear in $\vec{\ }$, which is of the order of the theory considered here. The change in the potential energy contains the usual thermal term linear in $\vec{R}_{\vec{q}}$ and an additional term, corresponding to the simultaneous absorption of a thermal phonon and a sound wave phonon. It is just this additional term which changes the equilibrium distribution from f_o to $\bar{f}$. We find that one may use just (70) to determine the collision integral except that now the energy conservation terms must be modified to read

$$\Omega\left\{\delta_{\vec{K}+\vec{q}} - \delta_{\vec{K}} - h\nu_q + m\vec{u}\cdot(\vec{v}_{\vec{K}} - \vec{v}_{\vec{K}'})\right\}\delta_{\vec{K},\vec{K}'-\vec{q}} - \kappa_i(\vec{K},\vec{K}') \tag{81a}$$

and

$$\Omega\left\{\delta_{\vec{K}-\vec{q}} - \delta_{\vec{K}} + h\nu_q + m\vec{u}\cdot(\vec{v}_{\vec{K}} - \vec{v}_{\vec{K}'})\right\}\delta_{\vec{K},\vec{K}'-\vec{q}} - \kappa_i(\vec{K},\vec{K}'), \tag{81b}$$

where

$$\kappa_i(K,K') = \frac{m}{A}(\vec{v}_{\vec{K}} - \vec{v}_{\vec{K}'})\cdot\vec{e}_i(\vec{r}_o) \tag{81c}$$

and

$$\vec{e}_i(\vec{r}_o) = iu_i\vec{q}_i e^{i\vec{q}_i\cdot\vec{r}_o} + \text{complex conjugate}, \tag{81d}$$

here $\vec{e}_i(\vec{r}_o)$ is the strain dyadic associated with the impressed wave and is proportional to the lattice displacement.

Substituting (14),

$$\bar{f} = \left[\exp\left\{\left(\frac{m}{2}|\vec{v} - \vec{u}|^2 - \delta_o - \xi_1\right)/\kappa_o T\right\} + 1\right]^{-1},$$

(81a) to (81d), and (71) into (70), we see that $\bar{f}$ is the equilibrium distribution function under the above modifications. Consider

the first term in (70) subject to the new momentum relation:

$$(N_q + 1)\bar{f}(\vec{r},\vec{v})\left\{1 - \bar{f}(\vec{r})\right\} - N_q\bar{f}(\vec{r})\left\{1 - \bar{f}(\vec{r}+\vec{r})\right\}$$

$$= N_q\bar{f}(\vec{r})\bar{f}(\vec{r}+\vec{r})\left[\exp\left\{\left(\frac{m}{2}|\vec{v}-\vec{u}|^2 - \varepsilon_0 - \xi_1 + h\nu_q\right)/kT_0\right\}\right.$$

$$\left. - \exp\left\{\left(\frac{m}{2}|\vec{v}' - \vec{u}|^2 - \varepsilon_0 - \xi_1\right)/kT_0\right\}\right]$$

$$= N_q\bar{f}(\vec{r})\bar{f}(\vec{r}+\vec{r})\exp\left\{\left(\frac{m}{2}|\vec{v} - \vec{u}|^2 - \varepsilon_0 - \xi_1 + h\nu_q\right)/kT_0\right\}$$

$$\times\left[1 - \exp\left\{\left(\frac{m}{2}|\vec{v}' - \vec{u}|^2 - \frac{m}{2}|\vec{v} - \vec{u}| - h\nu_q\right)/kT_0\right\}\right]$$

$$= 0,$$

since

$$\varepsilon_{\vec{K}'} = E_{\vec{K}} + h\nu_q - m\vec{u}\cdot(\vec{v}_{\vec{K}} - \vec{v}_{\vec{K}'})$$

or so

$$\frac{m}{2}|\vec{v}' - \vec{u}|^2 - \frac{m}{2}|\vec{v} - \vec{u}|^2 - h\nu_q$$

$$= \varepsilon_{\vec{K}'} - \varepsilon_{\vec{K}} - h\nu_q + m\vec{u}\cdot(\vec{v}_{\vec{K}} - \vec{v}_{\vec{K}'}) = 0.$$

Hence, the first term in (70) is zero. Similarly, the second term is also zero for $f = \bar{f}$.

Finally, if we use (73) in (70) and Boltzmann's results, the collision integral reduces to (74) with f_0 replaced by $\bar{f}$ and Ω' replaced by (81a) and (81b). Further, remembering that we are retaining only terms linear in u and noting that all the correction terms are proportional to u, we have to first order in u just the expression (74) as it stands, so that the collision integral is given by (80)].

Proceeding now, it is more convenient to take

$$f = f_o - f_1 u \frac{\partial f_o}{\partial E} , \qquad (84)$$

on the left-hand side of the Boltzmann equation. Comparing (15) and (73) with (82) we have

$$f_1 u = m\vec{v}.\vec{u} + \xi_1 + f_2 u . \qquad (84)$$

Substituting (81) and (83) into (80) results in

$$\left[\frac{\partial f}{\partial t}\right]_{coll} = \frac{1}{\Lambda \hbar^2 v} \frac{\partial f_o}{\partial E} \left(\frac{T}{\Theta}\right)^3 \int_{-\Theta/T}^{\Theta/T} \frac{z^2 dz}{|1-e^{-z}|} \frac{(e^{\eta}+1)}{(e^{\eta+z}+1)}$$

$$\times \left\{ m\vec{v}.\vec{u} \left(\frac{q\cos\chi_+}{\tau}\right) + f_1(\vec{K})u - \left\langle f_1(\vec{K}+\vec{q}_+)u \right\rangle_{\omega'} \right\}$$

$$= - m\vec{v}.\vec{u} \frac{\partial f_o}{\tau_1(v)} + u \frac{\partial f_o}{\partial E} I_1 , \text{ say,} \qquad (84)$$

where

$$I_1 = \frac{1}{\hbar^2 v \Lambda} \left(\frac{T}{\Theta}\right)^3 \int_{-\Theta/T}^{\Theta/T} \frac{z^2 dz}{|1-e^{-z}|} \frac{(e^{\eta}+1)}{(e^{\eta+z}+1)} \left\{ f_1(\vec{K}) - \left\langle f_1(\vec{K}+\vec{q}_+) \right\rangle_{\omega'} \right\} \qquad (85a)$$

and

$$\frac{1}{\tau_1(v)} = \frac{1}{\hbar^2 v \Lambda} \left(\frac{T}{\Theta}\right)^3 \int_{-\Theta/T}^{\Theta/T} \frac{z^2 dz}{|1-e^{-z}|} \frac{(e^{\eta}+1)}{(e^{\eta+z}+1)} \left\{ \frac{1}{2} \left[2\eta \left(\frac{T}{\Theta}\right)^2 z^2 - \ln \left| \frac{e^{z}-1}{z} \right|^2 \right] \right\}, \qquad (85b)$$

where

$$\eta = \frac{(6\pi^2)^{2/3} \hbar^2}{4ma^2} .$$

We note that (85b) is identical to the expression found for the relaxation time for electrical conductivity in equation (9.61) of Wilson, [19].

The complete Boltzmann equation can be written, using (83) and (84) in (16), in the form

$$(-i\omega + ikv_x)f_1 u \frac{\partial f_0}{\partial \varepsilon} + e\vec{\mathcal{E}}.\vec{v} \frac{\partial f_0}{\partial \varepsilon} = -\frac{m\vec{v}.\vec{u}}{\tau_1} \frac{\partial f_0}{\partial \varepsilon} + f_1 u \frac{\partial f_0}{\partial r} . \qquad (86)$$

We now expand f_1 in terms of spherical harmonics. For a bounded continuous function this is always possible and we assume that (86) has such a well behaved solution. Hence,

$$f_1(\vec{v}) = \sum_{\ell,m} v^\ell C_\ell^m(v) Y_\ell^m(\theta,\phi) , \qquad (87)$$

The next step is to substitute (87) into (86), multiply through by $Y_p^n(\theta,\phi)\sin\theta d\theta d\phi$, and then integrate over all angles.

This gives us the following integrals to evaluate:

$$\int Y_p^n(\theta,\phi) \sum_{\ell,m} v^\ell C_\ell^m(v) Y_\ell^m(\theta,\phi)\sin\theta d\theta d\phi$$

$$= \frac{2\pi\Delta_n}{(2p+1)} \frac{(p+n)!}{(p-n)!} v^p \delta_{\ell p}^n(v) , \qquad (88a)$$

where Δ_n is one for $n \neq 0$ and two for $n = 0$.

$$\int v_x Y_p^n(\theta,\phi)\sin\theta d\theta d\phi = \frac{2\pi\Delta_n}{(2p+1)} \frac{(p+n)!}{(p-n)!} v \delta_{p1}\delta_{n0} . \qquad (88b)$$

$$\int v_y Y_p^n(\theta,\phi)\sin\theta d\theta d\phi = \frac{2\pi\Delta_n}{(2p+1)} \frac{(p+n)!}{(p-n)!} v \delta_{p1}\delta_{n1} . \qquad (88c)$$

$$\int v_z Y_p^n(\theta,\phi) \sum_{\ell,m} v^\ell C_\ell^m(v) Y_\ell^m(\theta,\phi)\sin\theta d\theta d\phi$$

$$= \int Y_p^n(\theta,\phi) \sum_{\ell,m} v^{\ell+1} C_\ell^m(v) \left[\frac{(\ell+m)Y_{\ell-1}^m(\theta,\phi)}{(2\ell-1)} + \frac{(\ell-m-1)Y_{\ell+1}^m}{(2\ell-1)} \right]$$

$$\times \sin\theta d\theta d\phi .$$

$$= \frac{2\pi\Delta_n}{(2p+1)} \frac{(p+n)!}{(p-n)!} \left[\frac{(p-n)}{(2p-1)} v^p C_{p-1}^n(v) + \frac{(p+n+1)}{(2p+3)} v^{p+2} C_{p+1}^n(v) \right] . \qquad (88d)$$

Further, in the collision integral we may now carry out the integration over ω and this gives

$$\left\langle f_1(K+q_+) \right\rangle_\omega = \left\langle \sum_{\ell,m} v'^\ell C_\ell^m(v') Y_\ell^m(\theta',\phi') \right\rangle_\omega$$

$$= \sum_{\ell,m} v'^\ell C_\ell^m(v') \left\langle Y_\ell^m(\theta',\phi') \right\rangle_\omega$$

$$= \sum_{\ell,m} v'^\ell C_\ell^m(v') P_\ell(\cos\Omega) Y_\ell^m(\theta,\phi) . \qquad (88e)$$

The steps in (88e) follow since $v' = v\left[1 - \frac{E_0}{E}\right]^{1/2}$, is independent of ω and

$$\left\langle Y_\ell^m(\theta',\phi') \right\rangle_\omega = P_\ell(\cos\Omega) Y_\ell^m(\theta,\phi) . \qquad (88f)$$

In these expressions Ω is now the angle between $\vec{K}$ and $\vec{K}' = \vec{K} + \vec{q}_+$.

In this case

$$\cos\Omega = \left[1 + \frac{k_o''z}{2E} - \frac{D}{E}\left(\frac{\pi}{\mathcal{H}}\right)^2 z^2 \right] \left[1 + \frac{k_o''z}{E} \right]^{-1/2}$$

$$= \frac{v}{v'} \left[1 + \frac{k_o''z}{2E} - \frac{D}{E}\left(\frac{\pi}{\mathcal{H}}\right)^2 z^2 \right] . \qquad (88g)$$

Substituting (88a) to (88g) into (86) and dividing out the factor $\frac{2\pi\Delta_n}{(2p+1)} \frac{(p+n)!}{(p-n)!}$ the Boltzmann equation becomes,

$$- i\omega v^p C_p^n(v) + ik \left[\frac{(p-n)}{(2p-1)} v^p C_{p-1}^n(v) + \frac{(p+n+1)}{(2p+3)} v^{p+2} C_{p+1}^n(v) \right]$$

$$+\frac{ev}{u}\left[\mathcal{E}_x\delta_{p1}\delta_{n0}+\mathcal{E}_y\delta_{p1}\delta_{n1}+\mathcal{E}_z\delta_{p1}\delta_{n,-1}\right]=-\frac{mv}{\tau_1}\left[\delta_{p1}\delta_{n0}+\right.$$

$$\left.\delta_{p1}\delta_{n1}+\delta_{p1}\delta_{n,-1}\right]+\frac{1}{\Lambda\hbar^2v}\left(\frac{T}{\Theta}\right)^3\int_{\Theta/T}^{\Theta/T}\frac{z^2dz}{|1-e^{-z}|}\frac{(e^{\eta}+1)}{(e^{\eta+z}+1)}\left[v'^p C_p^n(v')\right.$$

$$\left.-v'^p C_p^n(v')P_p(\cos\Omega)\right]. \qquad (9a)$$

We can now notice several features of the Boltzmann equation in this form. Each p and n value gives a linear integral equation coupling three consecutive C_p^n's with the same n. Indeed we have an infinite set of coupled linear integral equations for each n. Further if we impose the conditions of no y- and z-components of current for the incident plane dilational wave moving in the x-direction, that is, $C_1^1 = C_1^{-1} = 0$, then we have all

$$C_p^{\pm1} = 0$$

Hence in this case we need only retain the set of equations for n = 0. Similarly, for a plane shear wave moving in the x-direction and polarized in the y-direction we need only consider the set of equations for n = 1. Thus we have: (i) for the incident plane dilational wave;

$$-i\omega C_p(v)+ik\left[\frac{p}{2p-1}C_{p-1}(v)+\frac{(p+1)}{(2p+3)}v^2C_{p+1}(v)\right]+\frac{e\mathcal{E}}{u}\delta_{p1}$$

$$=-\frac{m}{\tau_1}\delta_{p1}+\frac{1}{\Lambda\hbar^2v}\left(\frac{T}{\Theta}\right)^3\int_{\Theta/T}^{\Theta/T}\frac{z^2dz}{|1-e^{-z}|}\frac{(e^{\eta}+1)}{(e^{\eta+z}+1)}\left[C_p(v)-\right.$$

$$\left.\left(\frac{v'}{v}\right)^p P_p(\cos\Omega)C_p(v')\right], \qquad (9b)$$

where p = 0, 1, 2, 3, ... , and

(ii) for the incident plane shear waves;

$$-i\omega C_p(v) + ik\left[\frac{(p-1)}{(2p-1)}C_{p-1}(v) + \frac{(p+2)}{(2p+3)}v^2 C_{p+1}(v)\right] + \frac{e\mathcal{E}}{u}\delta_{p1}$$

$$= -\frac{m}{\tau_i}\delta_{p1} + \frac{1}{\Lambda\hbar^3 v}\left(\frac{m}{\hbar}\right)^3 \int_{\hbar/T}^{\hbar/r} \frac{z^2 dz}{|1-e^{-z}|} \frac{(e^{\eta}+1)}{(e^{\eta+z}+1)}\left[C_p(v) - \left(\frac{v'}{v}\right)^p P_p(\cos\Omega)C_p(v')\right], \quad (90b)$$

where $p = 1,2,3,\ldots$, and the index $n = 1$ has been suppressed in (90b).

To deal with the integrals appearing on the right hand sides of (90a) and (90b) we assume that each $C_p(v)$ is a very slowly varying function of energy so that, on integrating these expressions, multiplied by $\frac{\partial f_o}{\partial E}v^2 dv$, over all energies, we can set $C_p(v') \simeq C_p(v)$ in these integrals. We are then left with a temperature series through the $\left(\frac{v'}{v}\right)^p \times P_p(\cos\Omega)$ and since we are considering only very low temperatures we need only retain the lowest order temperature terms. The next step is to multiply (90a) and (90b) by $v^2\frac{\partial f_o}{\partial E}dv$ and integrate over all energies. This integration is considerably simplified by the fact that $\frac{\partial f_o}{\partial E}$ is almost an energy delta function.

We have now to evaluate integrals of the type:

$$\int_o^\infty v^{n+2} C_p(v)\frac{\partial f_o}{\partial E}dv = -\int_o^\infty \frac{v}{m}^{n+1} C_p(v)\delta(E-E_o)\,dE$$

$$= -\frac{v_o}{m}^{n+1} C_p(v_o), \quad (91a)$$

$$\int_o^\infty \frac{mv^2}{\tau_i}\frac{\partial f_o}{\partial E}dv = \int_o^\infty \frac{v^2 dv}{\Lambda\hbar^2 v^3}\left(\frac{m}{\hbar}\right)^3 \frac{\partial f_o}{\partial E}\int_{\hbar/T}^{\hbar/r} \frac{z^2 dz}{|1-e^{-z}|}\frac{(e^{\eta}+1)}{(e^{\eta+z}+1)}$$

$$\times \left\{ 2\Gamma\left(\frac{T}{\Theta}\right)^2 z^2 - k_o Tz \right\}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\Lambda \hbar^2 m v^2} \left(\frac{T}{\Theta}\right) \frac{\partial f_o}{\partial \eta} d\eta \int_{\Theta/T}^{\Theta/T} \frac{z^2 dz}{|1-e^{-z}|} \frac{(e^{\eta}+1)}{(e^{\eta+z}+1)} \left\{ 2\Gamma\left(\frac{T}{\Theta}\right)^2 z^2 - k_o Tz \right\}$$

$$= -\frac{1}{2\Lambda \hbar^2 E_o} \left(\frac{T}{\Theta}\right) \int_{-\infty}^{\infty} \int_{\Theta/T}^{\Theta/T} \frac{z^2 dz}{|1-e^{-z}|} \frac{e^{\eta} d\eta}{(e^{\eta}+1)(e^{\eta+z}+1)} \left\{ 2\Gamma\left(\frac{T}{\Theta}\right)^2 z^2 - k_o Tz \right\}$$

$$= -\frac{1}{2\Lambda \hbar^2 E_o} \left(\frac{T}{\Theta}\right)^3 \int_{\Theta/T}^{\Theta/T} \frac{z^3 dz}{|1-e^{-z}|(e^z-1)} \left\{ 2\Gamma\left(\frac{T}{\Theta}\right)^2 z^2 - k_o Tz \right\}$$

$$= -\frac{D}{\Lambda \hbar^2 E_o} \left(\frac{T}{\Theta}\right)^5 J_5\left(\frac{\Theta}{T}\right) . \qquad (91b)$$

$$\int_o^{\infty} \frac{1}{\Lambda \hbar^2 v} \left(\frac{T}{\Theta}\right)^3 \frac{\partial f_o}{\partial E} v^3 dv \int_{\Theta/T}^{\Theta/T} \frac{z^2 dz}{|1-e^{-z}|} \frac{(e^{\eta}+1)}{(e^{\eta+z}+1)} \left[C_p(v) - \left(\frac{v'}{v}\right)^p \right.$$

$$\left. \times P_p(\cos\Omega) C_p(v') \right]$$

$$= -\frac{1}{\Lambda \hbar^2 m} \left(\frac{T}{\Theta}\right)^3 \int_{-\infty}^{\infty} \int_{\Theta/T}^{\Theta/T} \frac{z^2 dz}{|1-e^{-z}|} \frac{e^{\eta} d\eta}{(e^{\eta}+1)(e^{\eta+z}+1)} \left[C_p(v) - \left(\frac{v'}{v}\right)^p \right.$$

$$\left. \times P_p(\cos\Omega) C_p(v') \right]$$

$$= -\frac{1}{\Lambda \hbar^2 m} \left(\frac{T}{\Theta}\right)^3 \int_{\Theta/T}^{\Theta/T} \frac{z^3 dz}{|1-e^{-z}|(e^z-1)} \left[C_p(v_o) \left\{ 1 - \left(\frac{v_o'}{v_o}\right)^p P_p(\cos\Omega) \right\} \right]$$

$$= -\frac{C_p(v_o)}{\Lambda \hbar^2 m} \left(\frac{T}{\Theta}\right)^3 \int_{\Theta/T}^{\Theta/T} \frac{z^3 dz}{|1-e^{-z}|(e^z-1)} \left[1 - P_p(\cos\Omega_o) \left\{ 1 + \frac{k_o Tz}{E_o} \right\} \right]$$

$$= -\frac{v_o C_p(v_o)}{m \tau_p}, \text{ say.} \qquad (91c)$$

The subscript zero denotes that the velocity and energy are to be evaluated at the Fermi surface. With the above approximations we have reduced the initial set of coupled linear integral equations to simply a set of linear algebraic equations, in the $C_p(v_o)$, which we may now solve. (91b) follows from the fact that only even powers of z in the curly brackets of the integrand give non-zero contributions to the integral and from the definition of J_n,

$$J_n = \int_0^{\Theta/T} \frac{z^n dz}{(1-e^{-z})(e^z-1)} .$$

We notice here that (96b) is simply

$$-\frac{.2D}{\Lambda \hbar^2 E_o}\left(\frac{T}{\Theta}\right)^5 J_5\left(\frac{\Theta}{T}\right) = -\frac{v_o}{\ell_1},$$

where ℓ_1 is the relaxation time, due to the electron-phonon interaction, which appears in the electrical conductivity at low temperatures.

Further, the expression in the square bracket of (91c) is a series in powers of z and T. First, as we have noted, we need retain only the even powers of z in this polynomial as the odd powers integrate out to zero. Secondly, since we are considering very low temperatures, we need retain only the lowest order term in the temperature. With this in mind, solutions for the two cases of low frequencies, $k\ell_1 << 1$, and of high frequencies, $k\ell_1 > 1$, may be obtained.

As a preliminary to sections 3.3 and 3.4 we make the following identities:

$$N = \left(\frac{2m^3}{h^3}\right)\int f d\vec{v} = N_o + \frac{3N_o}{3N_o} C_o(v_o)\, u = N_o + N_e u \,,$$

$$\vec{J}_e = \left(\frac{2em^3}{h^3}\right)\int \vec{v} f d\vec{v} = \frac{N_o eu}{m}\left[C_1^o(v_o)\underline{i} + C_1^1(v_o)\underline{j} + C_1^{-1}(v_o)\underline{k}\right],$$

$$N_o = \left(\frac{2m^3}{h^3}\right)\int v Y_2^o(\theta,\phi) f d\vec{v} = \frac{3N_o v_o}{5m} C_2^o(v_o) u \,,$$

$$N_1 = \left(\frac{2m^3}{h^3}\right)\int v Y_2^1(\theta,\phi) f d\vec{v} = \frac{9N_o v_o}{5m} C_2^o(v_o) u \,,$$

$$N_o = \left(\frac{2m^3}{h^3}\right)\int v Y_3^o(\theta,\phi) f d\vec{v} = \frac{3N_o u v_o^2}{7m} C_3^o(v_o) \,,$$

$$N = \left(\frac{2m^3}{h^3}\right)\int v Y_3^1(\theta,\phi) f d\vec{v} = \frac{18N_o u v_o^2}{7m} C_3^o(v_o) \,, \qquad (92)$$

3.2 THE RELAXATION TIMES, τ_p.

To obtain a complete picture of the collision integral, as calculated here, we rewrite (15) in the form,

$$\bar{f} = f_o - (\xi_1 + m\vec{v}.\vec{u})\frac{\partial f_o}{\partial E}$$

$$= \sum_p \bar{f}_p \quad , \; p = 0,1,2,\ldots,$$

where

$$\bar{f}_o = f_o - \xi_1 \frac{\partial f_o}{\partial E} \; ,$$

$$\bar{f}_1 = -m\vec{v}.\vec{u}\frac{\partial f_o}{\partial E} = -mv\frac{\partial f_o}{\partial E}\sum_{n=1}^{0,-1} u_n Y_1^n \; ,$$

and

$$\bar{f}_p = 0 \; , \text{ for } p = 2,3,\ldots$$

(82) may also be re-written in the form

$$f = f_o - \sum_{\rho,n} v^\rho C_\rho^n(v) Y_\rho^n(\theta,\phi) u \frac{\partial f_o}{\partial E}$$

$$= \sum f_\rho' \quad , \; p = 0,1,2,\ldots,$$

where

$$f_o' = f_o - C_o u \frac{\partial f_o}{\partial E} \; ,$$

and

$$f_\rho' = -\sum_{n=-\rho}^{\rho} v^\rho C_\rho^n(v) Y_\rho^n u \frac{\partial f_o}{\partial E} \quad , \quad p = 1,2,3,\ldots$$

The collision integral, (80), is equivalent to

$$\left[\frac{\partial f}{\partial t}\right]_{coll.} = -\frac{m\vec{v}.\vec{u}}{\tau_1}\frac{\partial f_o}{\partial E} + \sum_{\rho,n}\frac{1}{\tau_\rho} v^\rho C_\rho^n(v) Y_\rho^n(\theta,\phi) u \frac{\partial f_o}{\partial E}, \; p \neq 0$$

$$= - \sum_{\rho=1}^{\infty} \frac{f_\rho' - \bar{f}_\rho}{\tau_\rho}$$

$$= - \sum_{\rho=0}^{\infty} \frac{f_\rho' - \bar{f}_\rho}{\tau_\rho} \quad , \tag{93}$$

only if we make the identity

$$\xi_1 = C_0(v_0)u \quad , \tag{94}$$

This identity is the equivalent of (22), that is the equation of continuity.

We note that, in reality, it was necessary to carry out the integration over energy before a reduction of the collision integral into the form of (93) occurred, so that (93) is certainly not rigourous. However, (93) is equivalent to the preceding procedure where we set $\frac{\partial f_o}{\partial E} = - \delta(E - E_o)$ and take the $C_\rho(v)$'s as slowly varying function of energy. We also note that $C_o(v)$ need not be a constant but need only satisfy (94). Hence in this case we may consider each term in the expansion to relax towards its equilibrium value with a characteristic relaxation time, τ_ρ, and that there is no single relaxation time suitable to characterize the collision integral.

We shall now consider τ_1 and τ_2 in some detail. The calculation of τ_1 has been accomplished in (91b) and is

$$\frac{1}{\tau_1} = \frac{1}{\chi^2 v_o \Lambda} \left(\frac{T}{\Theta}\right)^3 \int_{\Theta/T}^{\Theta/T} \frac{z^3 dz}{|1 - e^{-z}|(e^z - 1)} \left[\frac{D}{D_o} \left(\frac{T}{\Theta}\right)^2 z^2 - k_o T z \right]$$

$$= \frac{2D}{\Lambda \chi^2 v_o D_o} \left(\frac{T}{\Theta}\right) J_5\left(\frac{\Theta}{T}\right). \tag{95}$$

In the same manner,

$$\frac{1}{\tau_2} = \frac{1}{\hbar^2 v_o \Lambda}\left(\frac{T}{\Theta}\right)^3 \int_{\Theta/T}^{\Theta/T} \frac{z^3\,dz}{|1-e^{-z}|\,(e^z-1)}\left[-\frac{k_o^2 T}{E_o} + \left\{\frac{3D}{E_o}\left(\frac{T}{\Theta}\right)^2 - \frac{3}{2}\left(\frac{k_o T}{2E_o}\right)^2 z^2 + \frac{3k_o^2 TD}{2E_o^2}\left(\frac{T}{\Theta}\right)^2 z^3 - \frac{3\pi^2}{2E_o^2}\left(\frac{T}{\Theta}\right)^4 z^4\right]\right.$$

$$= \frac{2}{\hbar^2 v_o \Lambda}\left(\frac{T}{\Theta}\right)^5\left[\left\{\frac{3D}{E_o}\left(\frac{T}{\Theta}\right)^2 - \frac{3}{2}\left(\frac{k_o T}{2E_o}\right)^2\right\} J_5\left(\frac{\Theta}{T}\right) - \frac{3\pi^2}{2E_o^2}\left(\frac{T}{\Theta}\right)^4 J_7\right]$$

$$= 3\,\frac{D}{\Lambda\hbar^2 v_o k_o}\left(\frac{T}{\Theta}\right)^5 J_5\left(\frac{\Theta}{T}\right)\left[\left\{1 - \frac{(k_o\Theta)^2}{4E_o D}\right\} - \frac{\pi^2}{2E_o}\left(\frac{T}{\Theta}\right)^2 \frac{J_7}{J_5}\right]$$

$$= \frac{3}{\tau_1}\left[1 - \frac{(k_o\Theta)^2}{4E_o D} - \frac{\pi^2}{2E_o}\left(\frac{T}{\Theta}\right)^2 \frac{J_7}{J_5}\right]$$

$$\simeq \frac{3}{\tau_1}, \qquad (96)$$

The second step follows since all odd powers z in the square brackets integrate to zero. The final reduction in (96) arises from the fact that we are considering very low temperatures, so that,

$$\frac{T}{\Theta} < 0.05 \quad \text{or} \quad T < 20^\circ K .$$

In metals, for the number of free electrons approximately equal to the number of atoms and where each unit cell contains only one or two atoms, then $D \simeq E_o$. For these conditions,

$$\frac{(k_o\Theta)^2}{4E_o D} \simeq 10^{-6},$$

since $\Theta \sim 300^\circ K$. Also,

$$\frac{\pi^2}{2E_o}\left(\frac{T}{\Theta}\right)^2 \frac{J_7}{J_5} \lesssim 0.05.$$

We remark that expression (96) for τ_2 is the same as that obtained by Steinberg, (1958b) [9], in his calculation of the viscosity of an electron gas resulting from the electron-phonon interaction.

Next, we shall consider the relaxation times in general. τ_p is defined in (91c) and is

$$\frac{1}{\tau_p} = \frac{1}{\Lambda \hbar^2 v_o} \left(\frac{T}{\Theta}\right)^3 \int_{\Theta/T}^{\Theta/T} \frac{z^3 dz}{|1-e^{-z}|(e^z-1)} \left[1 - P_p(\cos\Omega_o) \right.$$

$$\left. \times \left\{1 + \frac{k_o \hbar z}{m_o}\right\}^{p/2}\right],$$

where P_p is the pth Legendre polynomial and

$$\cos\Omega_o = \left[1 + \frac{k_o \hbar z}{2m_o} - \frac{p}{m_o}\left(\frac{T}{\Theta}\right)^2 z^2\right]\left[1 + \frac{k_o \hbar z}{m_o}\right]^{-1/2},$$

Since $\frac{k_o \Theta}{2m_o} \simeq 10^{-3}$, then $\frac{k_o \hbar z}{m_o} \lesssim 10^{-3}$ and we are able to neglect this factor throughout. Hence

$$\frac{1}{\tau_p} \simeq \frac{1}{\Lambda \hbar^2 v_o} \left(\frac{T}{\Theta}\right)^3 \int_{\Theta/T}^{\Theta/T} \frac{z^3 dz}{|1-e^{-z}|(e^z-1)} \left[1 - P_p(\cos\Omega_o)\right],$$

where we may now take

$$\cos\Omega_o = 1 - \frac{p}{m_o}\left(\frac{T}{\Theta}\right)^2 z^2$$

$$= 1 - \delta \quad , \text{ say.}$$

We may now write $P_p(\cos\Omega_o)$ as a power series in δ and obtain

$$P_p(\cos\Omega_o) \simeq P_p(1 - \delta) = 1 - \frac{p(p+1)}{2}\delta + \ldots$$

We now retain only the linear term in δ since the other terms contain higher powers of T and for very low temperatures these are expected to be decreasingly small. Hence

$$\frac{1}{\tau_p} = \frac{1}{\Lambda^2 v_o}\left(\frac{T}{\Theta}\right)^3 \int_{\Theta/T}^{\Theta/T} \frac{z^3 dz}{|1-e^{-z}|(e^z-1)}\left[\frac{p(p+1)}{2}\left(\frac{T}{\Theta}\right)^2 \frac{D}{E_o} z^2\right]$$

$$= \frac{p(p+1)}{2}\frac{1}{\tau_1},$$

that is,

$$\tau_p = \frac{2\tau_p}{p(p+1)}, \qquad (97)$$

for p = 1 and 2, (97) reduces, of course, to (95) and (96).

It is not immediately clear that we are justified in neglecting the higher order temperature terms in each τ_p. We note, however that these terms will be of the order

$$\left(\frac{T}{\Theta}\right)^{n-5} \frac{J_n(\Theta/T)}{J_5(\Theta/T)}, \quad n = 7,9,11,\ldots$$

and we have evaluated this quantity in table I for $T/\Theta = 0.05$. The last row indicates the order of magnitude of the neglected terms and we see that for all practical purposes we need only retain the first term, that is, for $T \lesssim 20^{\circ}K$. The values for $J_n(\Theta/T)$ were obtained from Wilson, [12] (p337).

We note that the relaxation times for higher order harmonics in the expansion become rapidly smaller than that for the first term. Hence, at low ultrasonic frequencies, we expect the higher order harmonics to be able to return to their equilibrium values and thus only the first few terms will be appreciably different from zero. In the

next two sections we consider this case. At higher frequencies, the higher order terms do not have time to reach their equilibrium values and hence they must be taken into account.

n	5	7	9	11	13	15	17
$J_n\left(\frac{H}{T}\right)$	1.25×10^{2}	5.08×10^{3}	3.46×10^{5}	3.99×10^{7}	6.23×10^{9}	1.31×10^{12}	3.56×10^{14}
$\frac{J_n}{J_5}$	1	40.6	2.91×10^{3}	3.19×10^{5}	4.98×10^{7}	1.05×10^{10}	2.85×10^{12}
$\left(\frac{T}{H}\right)$	1	2.5×10^{-3}	2.91×10^{-6}	1.56×10^{-8}	3.89×10^{-11}	9.71×10^{-14}	2.43×10^{-16}
$\left(\frac{T}{H}\right)^{n-5}\frac{J_n}{J_5}$	1	$\simeq 0.10$	$\simeq 0.02$	$\simeq 0.005$	$\simeq 0.002$	$\simeq 0.001$	$\simeq 0.0007$

TABLE I

Values of $\left(\frac{T}{H}\right)^{n-5}\frac{J_n}{J_5}$ for n = 5,7,...,17, $\frac{T}{H} = 0.05$

3.3 LOW FREQUENCY FLUID SHEAR WAVES.

Since the shear waves present somewhat simpler expressions than do the dilational waves, we shall treat this case first. For low frequencies, $k\ell_1 << 1$, we make the approximation that the series expansion, (87), is rapidly convergent and retain only the first two terms. As the solution unfolds, we see that this is a relatively good approximation in this frequency region. Rewriting (90b) with the aid of (91a), (91b), (91c), and (92) we have

$$(1 + i\omega\tau_p)C_p(v_o) - ik\tau_p\left[\frac{(p-1)\,C_{p-1}(v_o)}{(2p-1)} + \frac{(p+2)v_o^2 C_{p+1}(v_o)}{(2p+3)}\right]$$

$$= \left(m + \frac{e\mathcal{E}\tau_1}{u}\right)\delta_{p1}\,, \quad p = 1,2,3,\ldots \tag{98}$$

Hence

$$(1 + i\omega\tau_1)\frac{J_e}{e} - \frac{1}{3}ik\ell_1 N_1 = N_o u\left(1 + \frac{e\mathcal{E}\tau_1}{mu}\right) \tag{99}$$

and

$$(1 + i\omega\tau_2)N_1 - ik\ell_2\left\{\frac{3}{5}\left(\frac{J_e}{e}\right) + \frac{2}{5}N_1\right\} = 0, \tag{100}$$

where $\ell_1 = v_o\tau_1$ and $\ell_2 = v_o\tau_2$.

The solution now proceeds by neglecting N_1 compared to J_e/e in (100). The resulting equations can now be solved for J_e/e and N_1 and yield

$$(1 + i\omega\tau_2)N_1 = \frac{3}{5}ik\ell_2\left(\frac{J_e}{e}\right)$$

and

$$\left(\frac{J_e}{e}\right) = N_o u\left\{1 + \frac{e\mathcal{E}\tau_1}{mu}\right\}b\,, \tag{101}$$

where

$$b' = (1 + i\omega\tau_1) + \frac{k^2 \ell_1 \ell_2}{5(1 + i\omega\tau_2)},$$

We see that $N_1'J_e \sim ik\ell_1 << 1$. This process continues so that $N_1' << N_1$ and hence we may neglect N_1' compared to $J_e'e$ in (100).

The solution now follows exactly the same steps as those in section 2.3. Thus we have

$$e\mathcal{E} = \frac{mu\, k_o^2(1 - b')}{\tau_1}\left[k_o^2 b' + k^2 - \omega^2 c^2\right]$$

(analogous to equation (54)),

$$I_c = \frac{N_o mu}{\tau_1}(1 - b')\left[\frac{k^2}{k_o^2 b' + k^2}\right]$$

(analogous to equation (56)), and

$$\rho\ddot{s} = \rho\beta_s \frac{\partial^2 s}{\partial x^2} - \frac{N_o mu}{\tau_1}(1 - b')\frac{(k_o^2 + k^2)}{(k_o^2 b' + k^2)}$$

(analogous to equation (57)).

Since we are considering low frequencies, terms occur in the region $|k^2| << |k_o^2|$ and $|k^2| << |k_o^2 b'|$. Hence the equation of motion of the ions can be written,

$$\rho\ddot{s} = \rho\beta_s \frac{\partial^2 s}{\partial x^2} - \frac{N_o mu}{\tau_1}\left(\frac{1}{b'} - 1\right).$$

To obtain higher terms we must, of course, carry more terms in the series expansion for f_1. Again, as previously, since $k\ell_1 << 1$

$$\frac{1}{b'} - 1 = i\omega\tau_1 - \frac{\ell_1 \ell_2}{5}\left[\left\{(k_1^2 - k_2^2)(1 - \omega^2\tau_2^2) - 2\omega\tau_2 k_1 k_2 + \cdots\right\}\right.$$

$$- i\left\{2k_1k_2 + \omega\tau_2(k_1^2 - k_2^2)\right\}\left\{1 - \omega^2\tau_2^2 + \cdots\right\}\Big].$$

Therefore,

$$\alpha_s = 2k_2 = \frac{\omega m}{N\tau_1}\frac{\ell_1\ell_2}{5k_1\beta_s}\left[(k_1^2 - k_2^2)(1 - \omega^2\tau_2^2) - 2\omega\tau_2 k_1k_2 + \cdots\right]$$

$$= \frac{mv_o^2\omega\tau_2}{5N\beta_s k_1}\left[k_1^2 - 2\omega\tau_2 k_1k_2 + \cdots\right].$$

Hence,

$$\alpha_s = \frac{mv_o^2 k_1\omega\tau_2}{5N\beta_s} = \frac{mv_o^2\omega^2\tau_2}{5NV_s^3}\,. \qquad (102)$$

The dispersion relations are

$$\omega^2 = \beta_s(k_1^2 - k_2^2) - \frac{\omega m}{N\tau_1}\Big[\omega\tau_1 - \frac{\ell_1\ell_2}{5}\left\{2k_1k_2 + \omega\tau_2(k_1^2 - k_2^2)\right\}$$

$$\times\,(1 - \omega^2\tau_2^2 + \cdots)\Big],$$

which reduces to

$$V_s^2 = \frac{\omega^2}{k_1^2} = \frac{\beta_s}{(1 + m/m)}\left[1 + \frac{\alpha_s^2}{k_1^2} + \frac{mv_o^2}{5NV_s^2}\omega^2\tau_2^2 + \cdots\right]\left[1 - \frac{\alpha_s^2}{4k_1^2}\right]. \qquad (103)$$

We remark that (102) and (103) are precisely the results (62) and (63), section 2.3, with τ merely replaced by τ_2. That is, these expressions do not contain τ_1, the relaxation time appropriate for electrical conductivity. Of course, for low temperatures and/or the electron-phonon interaction alone we have seen that τ_2 is very nearly $\frac{1}{3}\tau_1$. In general, however, τ_2 will not be such a simple function of τ_1.

3.4 LOW FREQUENCY TRANSVERSE ELECTROMAGNETIC WAVES.

Here we deal with the most difficult case of oscillational waves. The difficulty arises due to the fact that the bulk motion of the electron gas must be taken into account. Letting $(\omega\tau_1) \ll 1$ with the aid of (91a), (91b), and (91c) we have

$$(1 + i\omega\tau_p)C_p(v_o) - ikv\tau_p\left[\frac{p}{(2p-1)}C_{p-1}(v_o) + \frac{(p+1)v^2}{(2p+3)}C_{p+1}(v_o)\right] = m\left(1 + \frac{e\mathcal{E}\tau_1}{mu}\right)\delta_{p1} \quad , \quad p = 1, 2, 3, \ldots \tag{101a}$$

For $p = 0$ we must use

$$i\omega C_o(v_o) - \frac{1}{3}ikv_o^2 C_1(v_o) = 0 \quad , \tag{101b}$$

This last equation is simply the equation of continuity for the electron or, equivalently, the collision integral must be zero when integrated over all velocities for $C_o(v)$. That is,

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{|1 - e^{-x}|(e^x - 1)}\left[C_o(v) - C_o(v')\right] = 0 \quad .$$

This relation is (actually) true for $C_o(v)$ independent of v, also for $C_o(v') = C_o(v)$ + only even functions of x.

Finally, using (92) we have

$$eu = \frac{1}{\omega}\left(\frac{J}{e}\right)_e \quad , \tag{101c}$$

$$(1 + i\omega\tau_1)\left(\frac{J}{e}\right)_e - \frac{1}{3}ikv_o^2 C_1(v_o) = nu\left(1 + \frac{\mathcal{E}\tau_1}{u}\right) \quad , \tag{101d}$$

$$(1 + i\omega\tau_2)C_o - \frac{1}{5}ikv_o^2 C_2\left\{2\left(\frac{J}{e}\right)_e + C_o\right\} = 0 \quad , \tag{101e}$$

Neglect now v_0 compared to J_e/e in (1.5c). The resulting equations have the solutions;

$$v_0 = \frac{k}{\omega}\left(\frac{J_e}{e}\right),$$

$$v_0 = \frac{2i\ \ell_2}{3(1 + i\omega\tau_2)}\left(\frac{J_e}{e}\right),$$

$$\frac{J_e}{e} = v_0 u\left(1 + \frac{e\mathcal{E}\tau_1}{mu}\right)\left[(1 + i\omega\tau_1) - \frac{1}{3}i\ell_1\left\{\frac{v_0}{\omega} + \frac{i\ \ell_2}{3(1 + i\omega\tau_2)}\right\}\right]^{-1}$$

$$= v_0 u\left(1 + \frac{e\mathcal{E}\tau_1}{mu}\right)\chi' \quad , \quad (2.3),$$

where

$$\frac{1}{\chi'} = (1 + i\omega\tau_1) - \frac{1}{3}i\ell_1\left\{\frac{v_0}{\omega} + \frac{i\ell_2}{3(1+i\omega\tau_2)}\right\}.$$

J_e/e in J_c/e is of the same form as that in (1.5c) with a complex relaxation time and the solution follows exactly the same line. Hence

$$e\mathcal{E} = \frac{mu}{\tau_1}\ \frac{(\chi' - 1)}{(1 + i\omega\gamma\chi'/\tau_1)},$$

where

$$\chi' = \frac{1}{\chi'} \quad ,$$

and

$$I_c = \frac{v_0 n}{\tau_1^2}\ \frac{i\omega\gamma(\chi' - 1)}{(1 + i\omega\gamma\chi'\tau_1)}.$$

When the quantity of motion ρu is taken, we have

$$\rho\dot{u} = \rho\beta_e\frac{\partial^2 n}{\partial x^2} - \frac{v_0 mn}{\tau_1}\ \frac{(\chi' - 1)(1 + i\gamma\omega\tau_1)}{(1 + i\gamma\omega\chi'\tau_1)}. \quad (2.4),$$

ir[illegible] we consider [illegible] frequencies $k\ell_1 << 1$ then we may again neglect the terms containing γ, that is, those proportional to the inverse square of the plasma frequencies. (106) then reduces to the simple form

$$\rho \ddot{\vec{n}} = \rho \beta_e^2 \frac{\partial^2 \vec{n}}{\partial x^2} - \frac{n_0 e m}{\varepsilon_1}(\chi' - 1) \,. \tag{106a}$$

For $k\ell_1 << 1$,

$$\chi' - 1 = \left[- \frac{2k_1 k_2 v_0 \ell_1}{3\omega} + \frac{4\ell_1 \ell_2 (k_1^2 - k_2^2 - 2\omega\tau_2 k_1 k_2)}{15(1+\omega^2\tau_2^2)} \right]$$

$$+ i\left[\omega\tau_1 - \frac{v_0 \ell_1}{3\omega}(k_1^2 - k_2^2) - \frac{4\ell_1\ell_2}{15}\frac{\{k_1 k_2 + \omega\tau_2(k_1^2 - k_2^2)\}}{(1+\omega^2\tau_2^2)}\right] . \tag{107}$$

Hence, equating real and imaginary parts in (107), we have

$$\gamma_2 = -\frac{2k_2 m v_0^2}{3\beta_e} + \frac{4 m v_0^2 \omega\tau_2}{15\beta_e^2 k_1}\frac{(k_1^2 - k_2^2 - 2\omega\tau_2 k_1 k_2)}{(1+\omega^2\tau_2^2)} \,.$$

Therefore,

$$\alpha_e = \gamma k_2 = \frac{4 m v_0^2 \omega^2 \tau_2}{15 v_e}\left[\beta_e\left(1 + \frac{m v_0^2}{3\beta_e}\right)\right]^{-1}$$

$$= \frac{4 m v_0^2 \omega^2 \tau_2}{15 v_e^3} \,. \tag{108}$$

And

$$\omega^2(1+\eta'^2) = \beta_e^2(k_1^2 - k_2^2)\left[1 + \frac{m v_0^2}{3\beta_e}\left\{1 + \frac{4}{5}\left(\omega^2\tau_2^2 + \omega\tau_2 \frac{k_2}{k_1}\right)\right\} + \cdots\right] .$$

Therefore,

$$\frac{\omega^2}{k_1^2} = v_e^2 = \frac{\beta_e(1 - \alpha_e^2(\ell_1^2))}{(1+\eta'^2)}\left[1 + \frac{m v_0^2}{3\beta_e}\left\{1 + \frac{4}{5}\left(\omega^2\tau_2^2 + \omega\tau_2 \frac{k_2}{k_1}\right)\right\} + \cdots\right]$$

$$= \beta_\ell \frac{(1 - \alpha_\ell^2/\beta_1^2)}{(1 + m_1^{-1})} \left[1 + \frac{mv^2}{3\beta_\ell} + \frac{\alpha_\ell}{\beta_1}\left(\omega\tau_2 + \frac{\alpha_\ell}{\beta_1}\right) + \cdots \right], \tag{108}$$

and

$$v_\ell^2 \simeq \frac{\beta_\ell}{(1 + m_1^{-1})} \left\{ 1 + \frac{mv^2}{3\beta_\ell} \right\}, \tag{109}$$

Again, we have just the same expressions for α_ℓ and v_ℓ^2 as those given in (45) and (46) with τ being replaced by τ_2. This means that the same relaxation time appears in the expression for the attenuation for both the shear waves and the dilatational waves. As already mentioned, the expression (96) for τ_2 is just the same as that in the expression for the shear viscosity of the electron gas calculated by Einbinder, (1958b) [9]. Also, our analysis shows that the electron-phonon interaction does not lead to a bulk viscosity of the electron gas. This result is in agreement with the experimental evidence of Pippard and Sommel, [4], which suggests that the electron gas has negligible, if any, bulk viscosity, (see (3)).

3.5 [illegible] QUASI-[illegible].

In section 3.3 and 3.4 we were able to obtain solutions by taking only the first few terms in the spherical harmonic expansion for f_1, (87). However, in the case of the higher frequencies $k\ell_1 >> 1$ we are no longer able to terminate the series at a satisfactory point. Once we attempt a solution by retaining all the terms. Further, in all events, it is necessary to obtain an expression only for J_o/o or, equivalently, $G_1(v_o)$, for both the incident shear wave and the incident dilatational wave. (93) has been written in the form

$$(1 + i\omega\tau_p)G_p(v_o) - ik\tau_p\left[\frac{(p-1)}{(2p-1)}G_{p-1}(v_o) + \frac{(p+2)v_o^2}{(2p+3)}G_{p+1}(v_o)\right]$$

$$= n\left(1 + \frac{e\mathcal{E}\tau_1}{m}\right)\delta_{p1} \quad , \quad p = 1,2,3,\ldots \tag{110}$$

(110) represents an infinite set of coupled linear algebraic equations in the $G_p(v_o)$'s and has as its solution a set of ratios of two infinite determinants. It is more convenient, however, to solve the following equations

$$-ia_p\frac{(p-1)}{(2p-1)}G_{p-1}(v_o) + G_p - ia_p\frac{(p+2)}{(2p+3)}G_{p+1}(v_o)$$

$$= \frac{n\alpha_1}{k\tau_1}\left(1 + \frac{e\mathcal{E}\tau_1}{m}\right)\delta_{p1} \quad , \quad p = 1,2,3,\ldots, \tag{111}$$

where

$$G_p(v_o) = v_o^p G_p(v_o),$$

and

$$a_p = \frac{kv_o\tau_p}{(1 + i\omega\tau_p)}.$$

We now use the method of determinantal ratios to write down the formal solution for this set of linear equations. In this case we have to deal with infinite determinant. and so we shall assume that the ratios converge to the actual solution of the problem. Further, we point out that an analytic solution over the complete range of frequencies has not been obtained and we have limited ourselves to the low and high frequency regions. However, as discussed later, numerical solutions can be obtained for the intermediate frequency range, but we shall not concern ourselves with these numerical solutions in this thesis.

(111) has the solution

$$\rho = D_\rho / D ,$$

where

$$D = \begin{vmatrix} 1 & -\frac{3}{5}ia_1 & 0 & 0 & 0 & 0 & \cdots \\ -\frac{1}{3}ia_2 & 1 & -\frac{4}{7}ia_2 & 0 & 0 & 0 & \cdots \\ 0 & -\frac{2}{5}ia_3 & 1 & -\frac{5}{9}ia_3 & 0 & 0 & \cdots \\ 0 & 0 & -\frac{3}{7}ia_4 & 1 & -\frac{6}{11}ia_4 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

and D_ρ is D with the ρth column replaced by $\frac{ma_1}{\chi z_1}\left(1 + \frac{e\mathcal{E}z_1}{mu}\right)$ in the first row and zeros elsewhere in the column. In this case we may write

$$\frac{J_e}{e} = \frac{I_o u}{mv_o}, \quad (v_o) = I_o u \left(1 + \frac{e\mathcal{E}z_1}{mu}\right)^{b''}, \qquad (112)$$

where

$$b'' = \frac{1}{mv_o}\left(1 + \frac{e\mathcal{E}z_1}{mu}\right)^{-1} = \frac{1}{mv_o}\left(1 + \frac{e\mathcal{E}z_1}{mu}\right)^{\frac{m}{e\mathcal{E}}} .$$

(112) is of exactly the same form as (52) and thus the solution can be carried out in the same manner. This results in

$$e\mathcal{E} = \frac{mu\, k_o^2 (1 - b'')}{\tau_1} \left(k_o^2 b'' + k^2 - \omega^2/c^2\right)$$

and

$$I_c = \frac{N_o mu}{\tau_1} (1 - b'') \left[\frac{k^2}{k_o^2 b'' + k^2} \right] .$$

Hence the equation of motion of the ions becomes

$$\rho \ddot{s} = \rho\beta_s \frac{\partial^2 s}{\partial x^2} - \frac{N_o mu}{\tau_1} (1-b'') \frac{(k_o^2 + k^2)}{(k_o^2 b'' + k^2)} . \qquad (113)$$

Again we have precisely an analogous form for the equations of motion as (57) in the case of an assumed relaxation time.

To evaluate the attenuation coefficient we must evaluate b'', or τ_1/τ, that is, the ratio of two infinite determinants. For $k\ell_1 << 1$, this ratio can be expressed in the form of a continued fraction which can be terminated at any point for any desired approximation. This is, of course, equivalent to terminating the series expansion of Δ in section 3.3, at any desired term, and leads to precisely the same results. When $k\ell_1 > 1$, this method no longer applies. However, it is found that the leading term in b'' can be determined quite easily. This term is sufficient to give the first order terms in the attenuation and the dispersion relations, but does not give the correction terms. For intermediate frequencies, $k\ell_1 \sim 1$, it seems that one must use a numerical method. Because τ_p and hence a_p decrease rapidly with increasing p, in practice, for numerical work we have always only to solve the ratio of two finite

determinants. For example, if $k\ell_1 = 1$, then $k\ell_{10} = \frac{k\ell_1}{110} \simeq 0.01$. Hence we expect that only a 10 x 10 matrix need to be used for this value of $k\ell_1$. We shall not attempt any numerical solutions in this thesis but shall assume that the solution carries smoothly from the low frequency limit to the high frequency limit.

To evaluate D_1/D and hence b'' for $k\ell_1 \gg 1$, we now write

$$D_1 = \frac{ma_1}{k\tau_1}\left(1 + \frac{e\ell\tau_1}{mu}\right) \begin{vmatrix} 1 & -\frac{3}{5}ia_1 & 0 & 0 & \cdots \\ 0 & 1 & -\frac{4}{7}ia_2 & 0 & \cdots \\ 0 & -\frac{2}{5}ia_3 & 1 & -\frac{5}{9}ia_3 & 0 \quad \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} .$$

Dividing a_p out of each row we have for $D_1(v_o)$

$$A_1(v_0) = \frac{ma_1}{k\tau_1}\left(1 + \frac{e\mathcal{E}\tau_1}{mu}\right) \frac{\begin{vmatrix} \frac{1}{a_1} & -i\alpha_{12} & 0 & 0 & \cdots \\ 0 & \frac{1}{a_2} & -i\alpha_{23} & 0 & \cdots \\ 0 & -i\alpha_{32} & \frac{1}{a_3} & -i\alpha_{34} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}}{\begin{vmatrix} \frac{1}{a_1} & -i\alpha_{12} & 0 & 0 & \cdots \\ -i\alpha_{21} & \frac{1}{a_2} & -i\alpha_{23} & 0 & \cdots \\ 0 & -i\alpha_{32} & \frac{1}{a_3} & -i\alpha_{34} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}},$$

where

$$\alpha_{i(i+1)} = \frac{(i+2)}{(2i+3)}, \text{ and } \alpha_{(i+1)i} = \frac{i}{2i+1}. \qquad (11)$$

From the form of the determinants we see that the denominator can be expanded in the form;

$$\delta_0 + \left\{\delta_{11}/a_1 + \delta_{12}/a_2 + \cdots\right\} + \left\{\delta_{21}/(a_1a_2) + \delta_{22}/(a_1a_3) + \cdots\right\} + \cdots$$

The numerator can also be expanded in the form;

$$\frac{1}{a_1}\left[\gamma_0 + \left\{\gamma_{22}/a_2 + \gamma_{23}/a_3 + \cdots\right\} + \cdots\right].$$

The δ's are infinite products of particular α'. Since only the leading term is relatively simple to evaluate and the rest are rather difficult, we shall restrict our calculations to the leading term alone, which allows us to calculate the attenuation coefficient but not the correction terms in the dispersion relations. Also although δ_0 by itself may not remain finite, all we need is the expres-

sion for $A_1(v_o)$ to remain finite. Hence we must evaluate γ_o/δ_o where this term is obtained by inspection from the determinants and

$$\delta_o = (-i\alpha_{12})(-i\alpha_{21})(-i\alpha_{34})(-i\alpha_{43})\ldots$$

and

$$\gamma_o = (-i\alpha_{23})(-i\alpha_{32})(-i\alpha_{45})(-i\alpha_{54})\ldots$$

Hence

$$\frac{\gamma_o}{\delta_o} = \frac{(\alpha_{23}\alpha_{32})(\alpha_{45}\alpha_{54})\ldots}{(\alpha_{12}\alpha_{21})(\alpha_{34}\alpha_{43})\ldots}$$

$$= \frac{\prod_{i=1}^{\infty}\{\alpha_{2i(2i+1)}\}\{\alpha_{(2i+1)2i}\}}{\prod_{i=1}^{\infty}\{\alpha_{(2i-1)2i}\}\{\alpha_{2i(2i-1)}\}},$$

We take the α's in pairs since we know that there is no factor of $\underline{i}$ appearing in this term. Furthermore, we see that there must be one more α appearing in the denominator than in the numerator as we start from the first row of D for δ_o and the second row of D_1 for γ_o. To evaluate this product subject to the above condition, we first evaluate the product for a finite number of terms and then take the limit as the number of terms goes to infinity together, that is, keeping the D and D_1 the same size. For a finite number of terms

$$\left(\frac{\gamma_o}{\delta_o}\right)_N = \prod_{i=1}^{N}\frac{\{\alpha_{2i(2i+1)}\alpha_{(2i+1)2i}\}}{\{\alpha_{(2i-1)2i}\alpha_{2i(2i-1)}\}}\,\frac{1}{\alpha_{(2N+1)2N}}$$

$$= \frac{(4N+1)}{2N}\prod_{i=1}^{N}\left[\frac{(2i+2)}{(4i+3)}\frac{(2i)}{(4i+1)}\frac{(4i+1)}{(2i+1)}\frac{(4i-1)}{(2i-1)}\right]$$

$$= \frac{(4N+1)}{2N}\prod_{i=1}^{N}\left[\frac{4i}{(2i-1)}\frac{(i+1)}{(2i+1)}\frac{(4i-1)}{(4i+3)}\right]$$

$$= 2^{2N}\frac{(4N+1)}{2N}\left[3\,\frac{1^2 2^2 3^2 4^2 \ldots (N-1)^2 N^2 (N+1)^2}{1^2 3^2 5^2 7^2 \ldots (2N-3)^2 (2N-1)^2 (2N+1)(4N+3)}\right]$$

$$= \frac{2^{2N}(4N+1)}{(4N+3)(2N)}\left[\frac{3(N!)^2}{\{(2N-1)!!\}^2}\frac{(N+1)}{(2N+1)}\right]$$

$$= \frac{3}{(2N)}\frac{(4N+1)}{(2N+1)}\frac{(N+1)}{(4N+3)}\left[\frac{2^{4N}(N!)^4}{(2N!)^2}\right].$$

We now apply Stirling's approximation in the form

$$N! = \sqrt{2\pi}\, N^{(N+1/2)} e^{-N}.$$

This gives

$$\left(\frac{\gamma_o}{\delta_o}\right)_N = \frac{3(N+1)}{2N}\frac{(4N+1)}{(2N+1)(4N+3)}\left[\frac{2^{2N}}{\sqrt{2\pi}}\frac{(2\pi)^{(2N+1)}}{(2N)^{(2N+1/2)}}\frac{N^{-2N}}{e^{-2N}}\right]^2$$

$$= \frac{3(N+1)(4N+1)}{2N(2N+1)(4N+3)}\left[(\pi N)^{1/2}\right]^2$$

$$= \frac{3\pi}{2}\frac{(N+1)}{(2N+1)}\frac{(4N+1)}{(4N+3)}$$

$$\xrightarrow{N\to\infty} \frac{3\pi}{4}. \qquad (115)$$

Therefore we have for the leading term in J_o/e, from (114) and (115),

$$\frac{J_o}{e} = \frac{N_o u A_1}{m v_o} = \frac{N_o u}{m v_o}\frac{m a_1}{k\tau_1}\left(1 + \frac{e\mathcal{E}\tau_1}{mu}\right)\frac{1}{a_1}\frac{\gamma_o}{\delta_o}$$

$$= N_o u\left(1 + \frac{e\mathcal{E}\tau_1}{mu}\right)\left(\frac{3\pi}{4k\ell_1} + \cdots\right). \qquad (115a)$$

That is,

$$b'' = \frac{3\pi}{4k\ell_1} + (\text{smaller terms}),$$

and indeed, $b'' << 1$.

We have now only to evaluate the attenuation coefficient. For $|k^2| << |k_o^2|$ and $|k^2| << |k_o^2 b''|$ in (113),

$$2k_2 = \alpha_s = \frac{\omega m}{N\tau_1/\beta_s k_1} \mathrm{Re}\left(\frac{1}{b''} - 1\right)$$

$$= \frac{\omega m}{N\tau_1/\beta_s k_1} \frac{4k_1 \ell_1}{3\pi} = \frac{4mv_o\omega}{3\pi N/\beta_s}$$

$$= \frac{4mv_o\omega}{3\pi N V_s^2} . \qquad (116)$$

This is precisely the limiting value found in the case in which we assumed the existence of a single relax tion time, (65).

For $k^2 >> |k_o^2 b''|$ and $k^2 >> |k_o^2|$, the situation is again the same as that found in section 2.3 and from (113),

$$\alpha_s = \frac{\omega m}{N\tau_1/\beta_s k_1} \mathrm{Re}\,(1 - b'')$$

$$= \frac{\omega m}{N\tau_1/\beta_s k_1}\left[1 - \frac{3\pi}{4k\ell_1} + \cdots\right] = \frac{m}{NV_s\tau_1} . \qquad (117)$$

Of particular interest is the fact that only τ_1 now appears in this first order expression for the attenuation coefficient.

3.6 HIGH FREQUENCY INCIDENT PLANE RELATIVISTIC LIMIT.

In this case, we use the same formalism as developed in section 3.5 and again restrict the calculations to an expression for the attenuation coefficient. The same comments hold here as in section 3.5 with the complications mentioned in section 3.4. We have (104b) and (104a)

$$C_o(v_o) - \frac{1}{3}\frac{kv_o}{\omega}C_1(v_o) = 0 , \quad p = 0 \tag{118a}$$

and

$$(1 + i\omega\tau_p)C_p(v_o) - ik\tau_p\left[\frac{p}{(2p-1)}C_{p-1}(v_o) + \frac{(p+1)}{(2p+3)}v_o^2 C_{p+1}(v_o)\right] = m\left(1 + \frac{e\mathcal{E}\tau_1}{mu}\right)\delta_{p1} , \quad p = 1,2,3,.. \tag{118b}$$

Again it is more convenient to set

$$A_p(v_o) = v_o^p C_p(v_o)$$

and solve the infinite set of linear equations for $A_p(v_o)$.

(118a) and (118b) result in

$$A_o(v_o) - \frac{1}{3}\frac{kv_o}{\omega}A_1(v_o) = 0 , \quad p = 0 \tag{119a}$$

and

$$-ia_p\frac{p}{(2p-1)}A_{p-1}(v_o) + A_p(v_o) - ia_p\frac{(p+1)}{(2p+3)}A_{p+1}(v_o) = \frac{ma_1}{k\tau_1}\left(1 + \frac{e\mathcal{E}\tau_1}{mu}\right)\delta_{p1} , \quad p = 1,2,3,.... \tag{119b}$$

We now have

$$D = \begin{vmatrix} 1 & -\frac{1}{3}ia_0 & 0 & 0 & 0 & \cdots \\ -ia_1 & 1 & -\frac{2}{5}ia_1 & 0 & 0 & \cdots \\ 0 & -\frac{2}{3}ia_2 & 1 & -\frac{3}{7}ia_2 & 0 & \cdots \\ 0 & 0 & -\frac{3}{5}ia_3 & 1 & -\frac{4}{9}ia_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix},$$

and D_1' is given by replacing the second column of D by all zeros except for the second row which has the value

$$\frac{ma_1}{k\tau_1}\left(1 + \frac{e\mathcal{E}\tau_1}{mu}\right).$$

In D, $ia_0 = \frac{kv_0}{\omega}$ and $a_\rho = \frac{k\ell_\rho}{(1 + i\omega\tau_\rho)}$, as before. And

$$A_1 = D_1'/D.$$

To evaluate A_1, first consider the ratio D'/D_1' which can be written

$$\frac{D'}{D_1'} = \frac{k\tau_1}{ma_1}\left(1 + \frac{e\mathcal{E}\tau_1}{mu}\right)^{-1}\left[\frac{1}{3}a_0a_1 + \frac{D'}{D_1'}\right],$$

where D' is the minor of the first element in D and D_1' is the minor of the first element in D_1, divided by $\frac{ma_1}{k\tau_1}\left(1+\frac{e\mathcal{E}\tau_1}{mu}\right)$.

We have now to evaluate D'/D_1'.

$$\frac{p'}{p'_1} = \frac{\begin{vmatrix} 1/a_1 & -i\alpha_{12} & 0 & 0 & 0 & \cdots \\ -i\alpha_{21} & 1/a_2 & -i\alpha_{23} & 0 & 0 & \cdots \\ 0 & -i\alpha_{32} & 1/a_3 & -i\alpha_{34} & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}}{\begin{vmatrix} 1/a_1 & -1\alpha_{12} & 0 & 0 & 0 & \cdots \\ 0 & 1/a_2 & -i\alpha_{23} & 0 & 0 & \cdots \\ 0 & -i\alpha_{32} & 1/a_3 & -i\alpha_{34} & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}} ,$$

where now

$$\alpha_{i(i+1)} = \frac{i+1}{2i+3} \quad \text{and} \quad \alpha_{(i+1)i} = \frac{i+1}{2i+1} .$$

The leading term in a series expansion of p'/p'_1 will now be $a_1 \delta_0/\gamma_0$ where

$$\delta_0 = (-i\alpha_{12})(-i\alpha_{21})(-i\alpha_{34})(-i\alpha_{43}) \cdots .$$

and

$$\gamma_0 = (-i\alpha_{23})(-i\alpha_{32})(-i\alpha_{45})(-i\alpha_{54}) \cdots .$$

Hence

$$\frac{\delta_0}{\gamma_0} = \frac{(\alpha_{12}\alpha_{21})(\alpha_{34}\alpha_{43})(\alpha_{56}\alpha_{65}) \cdots}{(\alpha_{23}\alpha_{32})(\alpha_{45}\alpha_{54})(\alpha_{67}\alpha_{76}) \cdots} .$$

Again we sum for a finite number of terms, using the knowledge that there are no factors of i left singly in this expression and that the product δ_0 contains one more α than does γ_0, since δ_0 is the term of a determinant one degree larger than that giving γ_0. We put this extra term at the beginning of the product, evaluate δ_0/γ_0

for a finite number of terms and take the limit as the number of terms goes to infinity together in δ_o and γ_o, as in section 3.5.

Now

$$\left(\frac{\delta_o}{\gamma_o}\right)_N = \prod_{i=1}^{N} \left\{\frac{\alpha_{(2i-1)2i}\,\alpha_{2i(2i-1)}}{\alpha_{2i(2i+1)}\,\alpha_{(2i+1)2i}}\right\} \alpha_{(2N+1)2N}$$

$$= \prod_{i=1}^{N} \left[\frac{(4i+3)}{(4i-1)}\left(\frac{2i}{2i+1}\right)^2\right] \frac{(2N+1)}{(4N+1)}$$

$$= \left[\frac{2^{2N}(N!)^2}{\{(2N+1)!!\}^2}\right]\left[\frac{(2N+1)(4N+3)}{3(4N+1)}\right]$$

$$= \left[\frac{2^{2N}(N!)^2}{(2N)!}\right]^2 \left[\frac{(2N+1)(4N+3)}{3(2N+1)^2(4N+1)}\right]$$

$$= \frac{\pi(4N+3)}{3(2N+1)(4N+1)},$$

on applying Stirling's approximation. In letting N tend to infinity, we have

$$\frac{\delta_o}{\gamma_o} = \frac{\pi}{6}. \tag{100}$$

Hence,

$$\frac{D}{D_1} = \frac{k\zeta_1}{m}\left(1 + \frac{e\mathcal{E}\zeta_1}{mu}\right)^{-1}\left(\frac{1}{3}a_o + \frac{\pi}{6}\right)^{-1}.$$

Therefore,

$$\frac{D_1}{D} = \frac{m}{k\zeta_1}\left(1 + \frac{e\mathcal{E}\zeta_1}{mu}\right)\left(\frac{1}{3}a_o + \frac{\pi}{6}\right),$$

and

$$\frac{J_e}{e} = \frac{N_o u A_1}{m v_o} = \frac{}{}{}_o u\left(1 + \frac{e\mathcal{E}\zeta_1}{mu}\right)\varepsilon'', \tag{101}$$

where

$$\frac{1}{\varepsilon''} = \frac{\pi k \ell_1}{6}\left(1 + \frac{2a_o}{\pi}\right) = \frac{\pi k \ell_1}{6}\left(1 + \frac{2kv_o}{i\pi\omega}\right). \tag{102}$$

We note that (121) is in precisely the same form as (115). The analysis just follows that in section 2.2. Neglecting the terms containing the plasma frequency, the equation of motion of the ions for $k\ell_1 >> 1$ is (analogous to that of section 2.2)

$$\rho \ddot{s} = \rho\beta_\ell \frac{\partial^2 s}{\partial x^2} - \frac{N_o m u}{L_1}\left(\frac{1}{g''} - 1\right). \qquad (122)$$

Using (121a) and equating imaginary parts in (122) we have

$$\alpha_\ell = 2k_2 = \frac{m\omega}{\sqrt{L_1\beta_\ell}\,k_1} \operatorname{Re}\left(\frac{1}{g''} - 1\right)$$

$$= \frac{\pi m v_o \omega}{6M\beta_\ell} - \alpha_\ell \frac{m v_o^2}{3M\beta_\ell}.$$

Therefore,

$$\alpha_\ell = \frac{\pi m v_o \omega}{6M}\left(\beta_\ell + \frac{m v_o^2}{3M}\right)^{-1}$$

$$= \frac{\pi m v_o \omega}{6M V_\ell^2}. \qquad (123)$$

We note again that (123) is just that, See (48), obtained when we assumed the existence of a single relaxation time. Also the term in the velocity due to the bulk modulus of the electron gas appears in this calculation and

$$V_\ell^2 \simeq \beta_\ell\left(1 + \frac{m v_o^2}{3M\beta_\ell}\right),$$

4. CALCULATION OF THE RELAXATION TIME: THE IMPURITY SCATTERING

When there are impurities in a metal, the electrons are not only scattered by the thermal motions of the ions but are also scattered by the impurities. The treatment of the collision integral arising from the impurities is similar to that for the electron-phonon interaction, (70). Also, the remainder of the analysis leading to the various expressions for the attenuation is quite analogous to that in the previous sections. Therefore we shall not give the details of this calculation and shall only state the results. The only difference is that we get the following relaxation times characteristic to each term in the expansion (87),

$$\frac{1}{\tau_p'} = \frac{(2\sigma + 1)^3 \Delta}{2\pi \varkappa_1^2 v} \int |H(\varkappa)|^2 \left[1 - P_p(\cos\Omega)\right] \varkappa \, d\varkappa , \qquad (124)$$

where $\varkappa = |\vec{K} - \vec{K}'| = K\left[2(1 - \cos\Omega)\right]^{1/2}$, Ω is the angle of scattering, that is, the angle between $\vec{K}$ and $\vec{K}'$, and

$$|H(\varkappa)|^2 = \frac{c}{\Delta^2 (2\sigma + 1)^3} \left| \int e^{i(\vec{K} - \vec{K}')\vec{r}} V(r) d\vec{r} \right|^2 \qquad (125)$$

is the absolute square of the matrix element for the transition of the electron from the state $\vec{K}$ into the state $\vec{K}'$. c is the concentration of the impurity atoms and V(r) is the perturbing potential due to one such impurity atom. In making this calculation we have assumed that the electrons can be described by plane wave functions, that there is only one type of impurity, and that the various impurity atoms are randomly distributed. We remark that expression (124) for $p = 2$ is the same as that obtained by Steinberg, (1958b) [9], in his calculations of the viscosity of the electron gas.

It is seen that if $|M(\chi)|^2$ is independent of the angle of scattering, that is χ, then the various τ_p''s are all equal and just τ_1'. In general, when $|M|^2$ depends on Ω then $\tau_p' \neq \tau_1'$. To get some idea of the ratios τ_p'/τ_1', we have made a calculation of $|M(\chi)|^2$ by taking for V(r) a screened Coulomb potential of the type

$$V(r) = \frac{Ae^{-\mu r}}{r}, \tag{126}$$

where $1/\mu$ is the range of the interaction and A is the strength constant. We find then,

$$|M(\chi)|^2 = \frac{c}{\Delta^2(2G+1)^3} \frac{16\pi^2\chi^2}{(\mu^2+\chi^2)^2}. \tag{127}$$

Therefore,

$$\frac{1}{\tau_p'} = \frac{1}{\tau_0'} \int_{-1}^{1} \frac{\{1 - P_p(\cos\Omega)\}}{(\beta - \cos\Omega)^2} d(\cos\Omega), \tag{128}$$

where

$$\frac{1}{\tau_0'} = \frac{2\pi c^2}{\Delta\hbar^2 v K^2} = \frac{\pi c A^2}{\Delta m v E}$$

and

$$\beta = 1 + \frac{\mu^2}{2K^2}. \tag{129}$$

Since $\beta > 1$, then we see from (128) that $\tau_p' \leq \tau_1'$, $p = 1,2,3,\ldots,$ for the interaction potential (126).* The case of particular interest is τ_1'/τ_2'.

* We may expect that τ_1'/τ_p' will be greater than unity and will have a similar qualitative behaviour so long as the scattering is predominantly in the forward direction and so long as it is a monotonically decreasing function of Ω.

pecifically, for p = 1 and p = 2, (108) reduces to

$$\frac{1}{\tau_1'} = \frac{1}{\tau_0'}\left[-\frac{2}{\beta+1} + \ln\left(\frac{\beta+1}{\beta-1}\right)\right], \tag{110}$$

$$\frac{1}{\tau_2'} = \frac{3}{\tau_0'}\left[-2 + \beta\ln\left(\frac{\beta+1}{\beta-1}\right)\right]. \tag{111}$$

e have plotted the ratio τ_1'/τ_2' in figure 4 against β, that is, effectively the range of the impurity potential. When β is large, this is for $\frac{1}{\mu} \ll 1$, $\frac{\tau_1'}{\tau_2'} = 1$. This is because the scattering becomes isotropic for this condition. On the other hand, for $\mu = 0$ $\frac{\tau_1'}{\tau_2'} = 3$.

In the case of a polyvalent impurity in a monovalent metal, the potential is approximately of the screened Coulomb type, as shown by Mott and Jones, (1936) [13]. For example, the screening constant μ is 0.55 x 10^{-8} cm. for such an impurity in copper. Remembering that for copper the number of electrons per unit volume is 8.4 x 10^{22} per c.c. then we have $\beta = 1.08$, evaluated at the Fermi surface and therefore $\frac{\tau_1'}{\tau_2'} = 1.9$.

e have seen that for $kl_1 \ll 1$ the attenuation coefficient is proportional to τ_2, (10) and (108). If we take only the impurities into account the analysis would be precisely the same and the new expressions will be simply (102) and (108) with τ_2 replaced by τ_2'. When the electrons are scattered both by impurities and by thermal phonons we shall assume that the effective relaxation times are given by

$$\frac{1}{\tau_p T} = \frac{1}{\tau_p} + \frac{1}{\tau_p'}, \qquad p = 1,2,3,\ldots \tag{112}$$

Actually, in our calculations (132) is not a separate assumption but follows from the various assumptions made in sections 3 and 4. For p = 1, (132) corresponds to Matthiessen's rule for the additivity of the reciprocals of the relaxation times in electrical conductivity due to the impurities and to the thermal motion of the ions. We now go on to discuss the results of this and the preceding sections in relation to the experimental data.

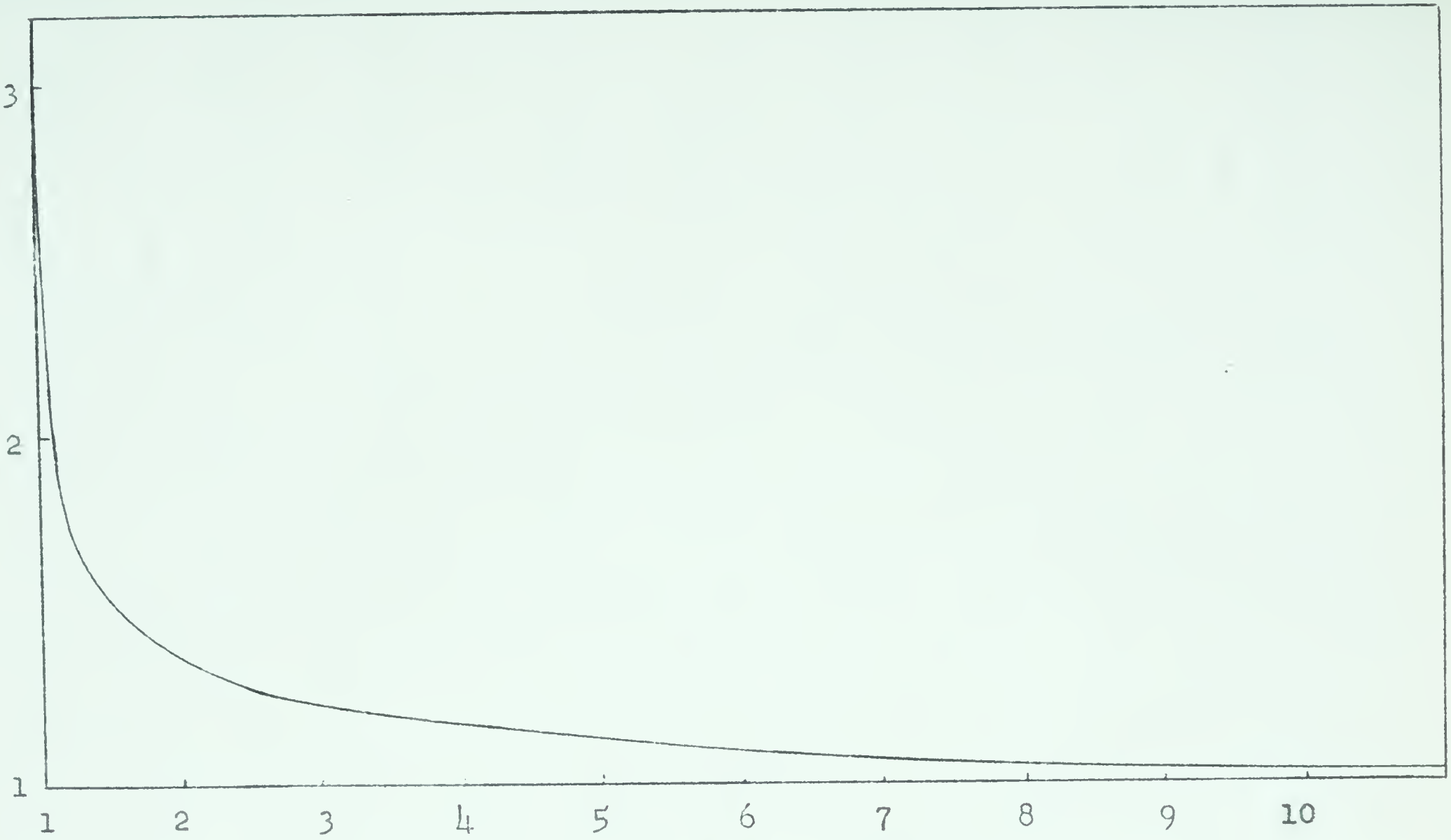


Figure 4.

Plot of τ_1'/τ_2' versus $\beta = 1 + \mu^2/2K_o^2$.

5. DISCUSSION

We should stress that our treatment of the attenuation problem in the previous sections is based on two main approximations: (1) The Sommerfeld free electron model of a metal is applicable and (2) the Fermi energy surface is spherical. In view of the above assumptions we cannot expect good quantitative agreement with the absolute values of the attenuation coefficient, particularly since quantitative measurements have been made on lead, tin, and aluminum only, [4]. Some measurements have also been made on copper but these have not been corrected for the energy losses inherent in the apparatus. We are also limited in verifying all the consequences of our work with experimental data for the following reasons: (1) The calculations of the relaxation times from the Boltzmann integral, at best, gives results to within an order of magnitude unity, as is well known from similar calculations of the electrical conductivity. (2) This difficulty could be overcome if measurements as a function of temperature were available for both the attenuation coefficient and the electrical conductivity on the same specimen. However, very few such measurements have been made. (3) Measurements have not been made, as a rule, over a sufficiently large frequency range at a fixed temperature so as to cover both the regions $kl_t \ll 1$ and $kl_t \gg 1$.

The preceding calculations indicate the following. First, for $kl_t \ll 1$ the relaxation time τ_a, appearing in the expressions for the attenuation coefficient, (102) and (108), is the same for both incident dilitional and incident shear waves. Referring to the last paragraph of section 4, we see that

$$\frac{1}{\tau_a} = \frac{1}{\tau_2} + \frac{1}{\tau_2'} = \frac{3}{\tau_1} + \frac{\delta}{\tau_1'} = \frac{1}{\tau_2^T}, \tag{133}$$

where τ_1 and τ_1' are the relaxation times appropriate for electrical conductivity, respectively due to the electron-phonon interaction and to the electron-impurity interaction. Moreover, δ is a constant of the order of magnitude unity. For a typical impurity potential such as the screened Coulomb type given in (126), δ lies between 1 and 3. When both electron-phonon scattering and electron-impurity scattering are present the relaxation time, τ_c, appropriate for electrical conductivity is given by (132),

$$\frac{1}{\tau_c} = \frac{1}{\tau_1^T} = \frac{1}{\tau_1} + \frac{1}{\tau_1'}. \tag{134}$$

Hence τ_a/τ_c will, in general, lie between 1 and 1/3.

A consequence of the fact that τ_a is the same for both the dilational and the shear waves is that the ratio α_s, (107), and α_ℓ, (108), is given by

$$\frac{\alpha_s}{\alpha_\ell} = \frac{3}{4}\left(\frac{V_\ell}{V_s}\right)^3\left(\frac{\omega_s}{\omega_\ell}\right)^2. \tag{135}$$

This ratio seems to be verified by experiment, [4], for single lead crystals in the normal state at 2°K and for $\omega_\ell = 26.5$MC/sec., $\alpha_\ell = 0.105$ nepers/cm. and $V_\ell = 2.39\text{-}10^5$ cm/sec. For $\omega_s = 10.1$MC/sec., $\alpha_s = 0.070$ nepers/cm. and $V_s = 1.27\text{x}10^5$ cm./sec. Hence $\alpha_s/\alpha_\ell = 0.67$ and $\frac{3}{4}\left(\frac{V_\ell}{V_s}\right)^3\left(\frac{\omega_s}{\omega_\ell}\right)^2 = 0.73$. Measurements of the attenuation of both dilational and shear waves on other substances are not yet available.

Secondly, a method of comparing our theoretical results on the relaxation times with experiment would be as follows. First, we

fit the theory to the experimental data at the low frequency limit by multiplying our expression (102) with an arbitrary constant, b. Figure 5 shows the plot of $\left[\alpha_t(\omega)/\omega\right]\left[\lim_{\omega\to\infty}\left\{\alpha_t(\omega)/\omega\right\}\right]^{-1}$ versus $k\ell_1$ for measurements on very pure tin crystals, [4]. In Pippard's calculation, where τ is identified as τ_c, the weighting factor is about 1/2, resulting in the curve indicated. In our case, when we take $\tau_a = \tau_c/3$, the weighting factor is about 3/2* and hence our high frequency limiting value will be just three times that of Pippard. τ_c, and hence $k\ell_1$, is determined by electrical conductivity measurements. Our limiting value of the dilational attenuation at high frequencies is much closer to the experimental value than Pippard's. If we take the impurities into account, then it is clear from (133) and (134) that τ_a lies between τ_c and $\tau_c/3$. This would improve the agreement with experiment.

A further consequence of (133) and (134) is that the ratio τ_a/τ_c is temperature dependent. Hence the weighting factor, b, will depend upon the temperature and so also the limiting value at high frequencies of the ratio plotted in figure 5. On the basis of our work this limiting value will decrease with decreasing temperature.

We note that shear waves behave somewhat differently than do the dilational waves. If we fit the attenuation for $k\ell_1 << 1$, as for the case of dilational waves, the value for the attenuation at $k\ell_1 \sim 1$ will again be about 2.5 times the value obtained by setting

---

*Measurements by Wilson, (1939) [14], give values of this weighting factor, b, of 3 for tin and 4.5 for aluminum.

$\tau_a = \tau_c$, throughout. For $k\ell_1 >> 1$ and $\omega\tau_1 >> 1$, however, the attenuation approaches the limiting value $\frac{m}{MV_s\tau_c}$ which will be about three times that expected for $\tau_a = \tau_c$. It should be noted that to obtain the condition $\omega\tau_1 >> 1$ requires much higher frequencies than have yet been obtained experimentally.

Throughout, we have considered processes which contribute both to the attenuation and to the electrical resistance. There is, however, one process, namely, the electron-electron collision, which makes no contribution to the electrical resistance since the total electron momentum is conserved in these collisions. On the other hand, these collisions will contribute to attenuation just as the atomic or molecular collisions contribute to viscosity. In normal metals, because of the strong screening, the electron-electron interaction will not be of much importance. If they are important, as may be the case below $1^{\circ}K$, then we will have

$$\frac{1}{\tau_a} = \frac{3}{\tau_1} + \frac{\delta}{\tau_1'} + \frac{1}{\tau_{ee}}, \tag{1-b}$$

where τ_{ee} is the relaxation time appropriate for electron-electron collisions and is expected to be proportional to T^{-2} for a degenerate Fermi gas. $1/\alpha$ will then have the form

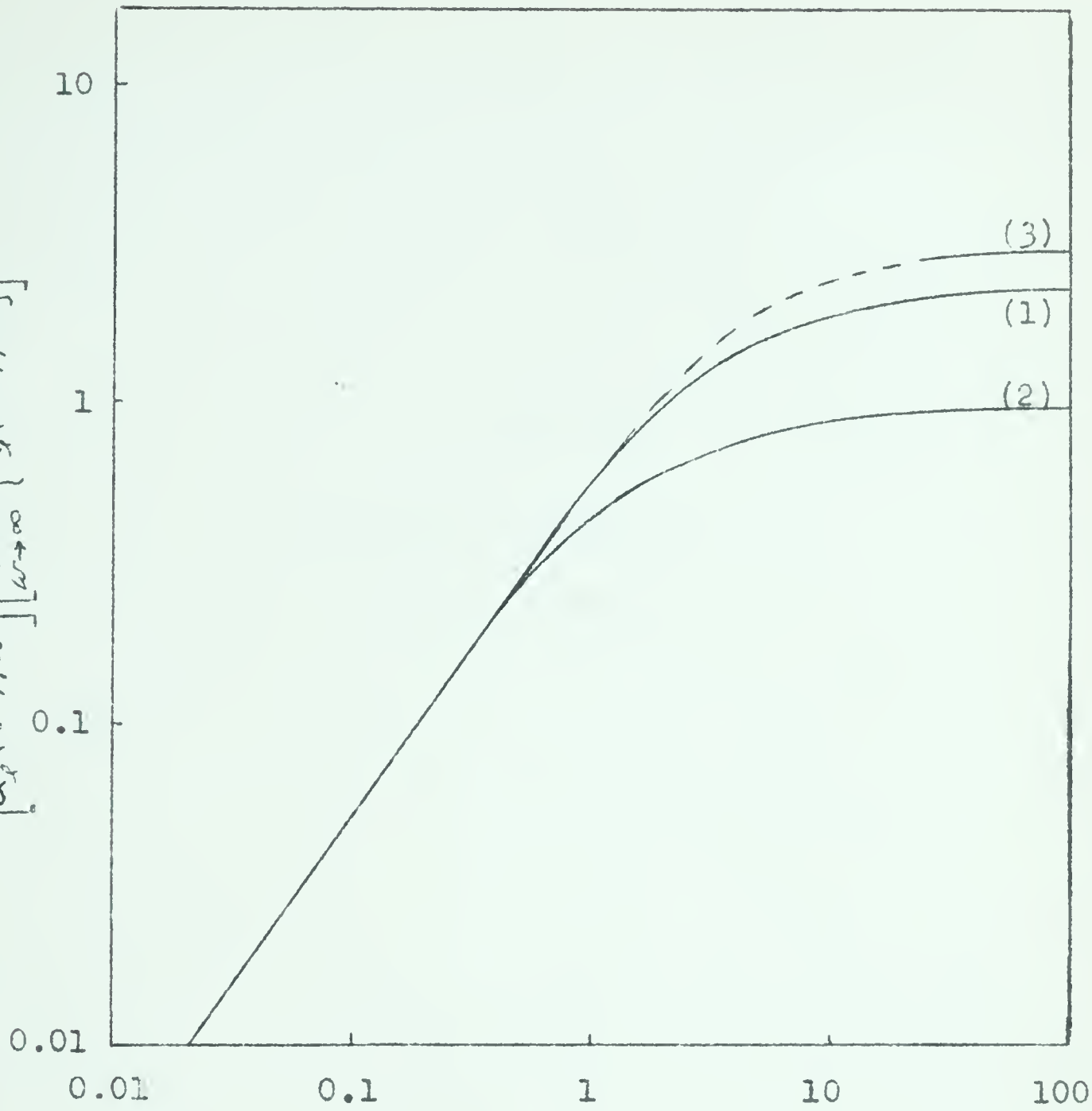
$$\frac{1}{\alpha} = A + BT^2 + CT^5, \tag{1-c}$$

whereas, the electrical resistivity, ρ, would be

$$\rho = A' + C'T^5. \tag{1-c}$$

Hence, by plotting $1/\alpha$ and ρ for the same specimen one may obtain information concerning τ_{ee}.

Although, as already mentioned, the electron-electron collisions are not likely to be of importance in metals, they may be important in some semi-metals and semi-conductors. The attenuation measurements may then be used to obtain information concerning the relaxation times for the electron-electron collisions.



Value of $k\ell_1$.

(1) Measured value for very pure tin at 60 megacycles.

(2) Pippard's theoretical results.

(3) Our theoretical results.

Figure 5.

Plot of $\left[\alpha_p(\omega)/\omega\right]\left[\lim_{\omega\to\infty}\left\{\alpha_p(\omega)/\omega\right\}\right]^{-1}$ versus $k\ell_1$.

PART II

ULTRASONIC ATTENUATION IN POLYCRYSTALS

6. INTRODUCTION.

As we have mentioned in the general introduction, Mason and McSkimin, (1947) [15] , (1948) [16] , have shown that one of the mechanisms through which attenuation arises in polycrystals is the scattering of sound waves by the grains. The scattering arises from the fact that the effective elastic constants relevant for the propagation of these waves differ from grain to grain because of the different orientations of the crystallographic axes of each grain with respect to a coordinate system fixed in space. In the frequency range for which the wavelength is large compared to the grain size Mason and McSkimin find that the amplitude attenuation coefficient, α_a . for both dilational and shear waves in polycrystalline aluminum and magnesium, may be written as

$$\alpha_a = D_1 f + D_2 f^4 . \qquad (1)$$

The first term is attributed to an elastic hysteresis, for which the precise mechanism is not yet known, and the second term to scattering. In this part of the thesis we shall deal with a calculation of D_2.

When a plane dilational wave is incident on a region of matter whose elastic constants or density differ from those of the surrounding bulk matter, both dilational and shear secondary waves are, in general, produced. If the shear modulus is zero, as for a fluid, only the former are produced. Mason and McSkimin realized that in a solid both types of scattered waves are produced. However in analysing their experimental results, they used a formula calculated by Rayleigh which refers to the scattering of sound waves in a fluid medium.

The value of D_2 obtained is several times less than the observed value for polycrystalline aluminum.

In the case of incident shear waves, Mason and McSkimin use a formula which is written down in analogy to Rayleigh's formula for the scattering of light due to inhomogeneities in the refractive index of the propagating medium. The variations in the refractive index of the medium arise out of density variations. We note that the scattering of light is due solely to the density variations, as evidenced by the angular distribution of the scattered intensity. On the other hand, in solids, the *sound* scattering is primarily due to the variations in the elastic constants. However, by identifying ρ, the density, in Rayleigh's formula with μ, the shear elastic constant, they obtain a value for D_2 which is about the same as the experimental value for aluminum.

Since the formulae used by Mason and McSkimin are such as to have incident plane dilational waves scattered only into dilational waves and incident shear waves scattered only into shear waves, Bhatia, (1959) [17], has improved on their calculation. His work is based on the following assumptions: (a) The grain size is small compared to the wavelength of the incident wave and that all the grains are nearly the same size. (b) The changes in the *effective* elastic constants and in the density of the medium from grain to grain are small, so that they may be treated as a perturbation and multiple scattering can be neglected and (c) The grains are randomly oriented, so that the bulk material may be treated as elastically isotropic and characterized by Lame's two elastic constants, λ_o and μ_o. A further simplification was made,

this being that the effective elastic properties of each grain are described by just two Lame's elastic constants, λ and μ, where $\lambda - \lambda_o << \lambda_o$ and $\mu - \mu_o << \mu_o$. His solution consists now of solving for the scattered energy from a single grain, summing over the scattered energy from all the grains in an element of volume, and hence obtaining a value for the attenuation coefficient. The main result of his work is that, for both incident dilational and shear waves, more than 90% of the scattered energy is carried away in the secondary shear waves. The attenuation coefficients obtained now are about twice too large for the incident dilational waves and only slightly larger for the incident shear waves than the experimental values. The theoretical ratio of the two attenuation coefficients is about six, agreeing reasonably well with the experimental value of 13.5 for aluminum, but not with that of unity for magnesium, [15].

Actually, an elastically anisotropic grain has several independent elastic constants, their number depending upon the crystalline symmetry of the grain. As a consequence, there are, in general, twenty-one, (not necessarily all independent), non-vanishing elastic constants, C'_{ij}, referred to a set of orthogonal axes, the laboratory coordinate system, not coincident with the crystal axes of the grain. This part of the thesis takes the existence of these elastic constants into account for the attenuation problem. Our assumptions are just those of Bhatia's with (c) modified so that all the elastic constants of the individual grains are retained. The solution follows the same pattern as his. We also assume that the elastic anisotropy is small.

In section 7 we formulate the general equations of motion

for an isotropic bulk material containing a single grain and solve for the scattering amplitude. These solutions are valid for polycrystals containing grains of any crystal symmetries. In section 8 we connect the expressions for the energy scattered by a single grain to the attenuation coefficient of the sound wave. The resulting expressions for attenuation involves certain averages and mean deviation averages. These averages are computed in Appendix I for crystals of orthorhombic symmetry. Crystals of higher symmetries are now simply special cases. Finally, in section 9 we discuss our results in relation to previous theoretical work and the experimental observations.*

---

*The results contained in this part of the thesis have appeared under the title "Scattering of High Frequency Sound Waves in Polycrystalline Materials" in Jour. Acous. Soc. Am. 31, 1140 (1959).

7. FORMULATION AND SOLUTION FOR THE SCATTERED WAVE FROM A SINGLE INHOMOGENEITY.

The equations of motion for a solid, in the absence of body forces are:

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial T_1}{\partial x} + \frac{\partial T_6}{\partial y} + \frac{\partial T_5}{\partial z},$$

$$\rho \frac{\partial^2 v}{\partial t^2} = \frac{\partial T_6}{\partial x} + \frac{\partial T_2}{\partial y} + \frac{\partial T_4}{\partial z}, \qquad (2)$$

$$\rho \frac{\partial^2 w}{\partial t^2} = \frac{\partial T_5}{\partial x} + \frac{\partial T_4}{\partial y} + \frac{\partial T_3}{\partial z},$$

where ρ is the density, u, v, and w are the Cartesian components of the displacement, and the T_i's are the components of the stress tensor. For small displacements it is a consequence of Hooke's Law that the stresses are proportional to the strains and hence we have

$$T_i = C_{ij} S_j, \qquad (3)$$

where we use the convention that repeated indices are summed. The C_{ij} are the elastic constants, and the S_j's are the stresses defined by

$$S_1 = \frac{\partial u}{\partial x}, \quad S_2 = \frac{\partial v}{\partial y}, \quad S_3 = \frac{\partial w}{\partial z},$$

$$S_4 = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, \quad S_5 = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}, \quad S_6 = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}. \qquad (4)$$

It is a consequence of the fact that the internal energy is a perfect differential that $C_{ij} = C_{ji}$ and hence there are at most 21 independent elastic constants, decreasing in number (to two for an isotropic crystal), as the crystal symmetry increases.

7.1 THE EQUATIONS OF MOTION FOR THE BULK MATERIAL.

In general, one denotes the elastic constants by C_{ij} when the axes of a fixed coordinate system coincide with the crystallographic axes of the single crystal. Usually, this situation is not accomplished and the single crystal axes are oriented with respect to the fixed axes. In this event, the equations of motion in the fixed coordinate system are given by (2) with the C_{ij}'s replace by C'_{ij}. the transformed elastic constants.* The C'_{ij}'s depend only on the C_{ij}'s and on the angles between the fixed axes and the crystallographic axes. These angles may be described in terms of the Euler angles for the rotation of the crystal axes from coincidence with the fixed axes to their actual position.

The bulk material is composed of small grains arbitrarily (or randomly) oriented. Since we are considering frequencies such that the wavelengths of the ultrasonic waves are large compared to the dimensions of the individual grains, we assume that the zero order elastic constants for the bulk material are the average values of the C'_{ij}'s over all random orientations. This results in the bulk material being described by

$$C_o = \langle C'_{11} \rangle = \langle C'_{22} \rangle = \langle C'_{33} \rangle$$

$$= \frac{1}{3}(C_{11} + C_{22} + C_{33}) - \frac{2P}{15} ,$$

$$\mu_o = \langle C'_{44} \rangle = \langle C'_{55} \rangle = \langle C'_{66} \rangle$$

$$= \frac{1}{3}(C_{44} + C_{55} + C_{66}) + \frac{1P}{15} , \qquad (5)$$

*The transformed elastic constants, the Euler angles, and methods of averaging are explained fully in Appendix I and the results only shall be stated here and in the following sections.

$$\lambda_o = \langle C'_{12} \rangle = \langle C'_{13} \rangle = \langle C'_{23} \rangle$$

$$= \frac{1}{3}(C_{12} + C_{13} + C_{23}) + \frac{1P}{15},$$

and $\langle C'_{ij} \rangle = \langle C'_{ji} \rangle = 0$, for i = 1,2,3 and j = 4,5,6,

where $P = (C_{11} + C_{22} + C_{33}) - (C_{12} + C_{13} + C_{23}) - 2(C_{44} + C_{55} + C_{66})$.

That is, we consider the host material as an isotropic medium with the elastic constants given by (5) and the equations of motion given by (2). We denote the solutions to these equations of motion by u_o, v_o, and w_o.

We shall assume that u,v, w and u_o, v_o, w_o are proportional to a simple harmonic function of time, exp(iωt), where $\omega = 2\pi f$ is the angular frequency of the incident wave and u, v, and w are the solutions to the generalized equations of motion. The time dependence will be suppressed except where noted.

We may now define the propagation constants, $(= 2\pi/\Lambda)$, k and h for the dilational and shear waves, respectively, as

$$k^2 = \frac{\omega^2 \rho_o}{C_o} = \frac{\omega^2}{V_l^2}, \quad h^2 = \frac{\omega^2 \rho_o}{\mu_o} = \frac{\omega^2}{V_s^2}, \tag{6}$$

where V_l and V_s are assumed* to be the zero order velocities of the dilational and shear waves in the bulk material.

* The values $V_l = \sqrt{\frac{C_o}{\rho_o}}$ and $V_s = \sqrt{\frac{\mu_o}{\rho_o}}$ are expected to be in agreement with observed values to within only about 10% because of the extreme simplications made here.

7.2. THE EQUATIONS OF MOTION FOR THE INHOMOGENEITY.

We consider now a single small grain arbitrarily oriented in this bulk material to be impinged upon by an incident wave. In the inhomogeneity the equations of motion are given by (4) where we must, of course, use the transformed elastic constants. These we write in the form

$$C'_{ij} = C^{o}_{ij} + \delta C_{ij}, \tag{7a}$$

where the C^{o}_{ij}'s are the average values given by (5) and where we shall assume that the δC_{ij}'s are small quantities, that is, they may be treated as a perturbation. We also assume that the solutions of the equations of motion can be written

$$u' = u_c + \delta u, \quad v' = v_o + \delta v, \quad w' = w_o + \delta w. \tag{7b}$$

Hence we may also write

$$S'_j = S^{o}_j + \delta S_j. \tag{7c}$$

Substituting (5), (7a), (7b), and (7c) into (2), retaining only linear terms in all the small quantities, because of our assumption of small anisotropy, that is, in all the δ's, and making use of the fact that u_c, v_o, and w_c are solutions to the average equations of motion, we obtain

$$\rho_o \omega^2 \delta u + \frac{\partial}{\partial x}\left[C_o \delta S_1 + \lambda_c(\delta S_2 + \delta S_3) \right] + \frac{\partial}{\partial y}\left[\mu_o \delta S_6 \right]$$

$$+ \frac{\partial}{\partial z}\left[\mu_c \delta S_5 \right] = -\left[\delta\rho\, \omega^2 u_o + H_x \right], \tag{8a}$$

$$\rho_o \omega^2 \delta v + \frac{\partial}{\partial x}\left[\mu_o \delta S_6\right] + \frac{\partial}{\partial y}\left[C_o \delta S_2 + \lambda_o(\delta S_1 + \delta S_3)\right]$$

$$+ \frac{\partial}{\partial z}\left[\mu_o \delta S_4\right] = -\left[\delta\rho\omega^2 v_o + H_y\right], \tag{8b}$$

$$\rho_o \omega^2 \delta w + \frac{\partial}{\partial x}\left[\mu_o \delta S_5\right] + \frac{\partial}{\partial y}\left[\mu_o \delta S_4\right] + \frac{\partial}{\partial z}\left[C_o \delta S_3\right.$$

$$\left. + \lambda_o(\delta S_1 + \delta S_3)\right] = -\left[\delta\rho\omega^2 w_o + H_z\right], \tag{8c}$$

where

$$H_x = \frac{\partial}{\partial x}(\delta C_{1j} S_j^o) + \frac{\partial}{\partial y}(\delta C_{6j} S_j^o) + \frac{\partial}{\partial z}(\delta C_{5j} S_j^o),$$

$$H_y = \frac{\partial}{\partial x}(\delta C_{6j} S_j^o) + \frac{\partial}{\partial y}(\delta C_{2j} S_j^o) + \frac{\partial}{\partial z}(\delta C_{4j} S_j^o), \tag{8d}$$

$$H_z = \frac{\partial}{\partial x}(\delta C_{5j} S_j^o) + \frac{\partial}{\partial y}(\delta C_{4j} S_j^o) + \frac{\partial}{\partial z}(\delta C_{3j} S_j^o),$$

summed over j. We note that the left hand side contains only the perturbed portion of the wave and the average values of the elastic constants and that the right hand side is determined completely by the initial wave and the inhomogeneity.

Differentiating each of (8) by x, y, and z, respectively, and adding we have, after making use of (4) and (6), and setting

$$\vec{s} = u\underline{i} + v\underline{j} + w\underline{k},$$

and

$$\vec{H} = H_x\underline{i} + H_y\underline{j} + H_z\underline{k},$$

$$\rho_o \omega^2(\vec{\nabla}\cdot\delta\vec{s}) + C_o\nabla^2(\vec{\nabla}\cdot\delta\vec{s}) = -\left[\omega^2\vec{\nabla}\cdot(\delta\rho\vec{s}_o) + \vec{\nabla}\cdot\vec{\Pi}\right],$$

or $$\nabla^2(\vec{\nabla}\cdot\delta\vec{s}) + k^2(\vec{\nabla}\cdot\delta\vec{s}) = -\frac{1}{C_o}\vec{\nabla}\cdot\left[\delta\rho\omega^2\vec{s}_o + \vec{\Pi}\right]$$

$$= -\vec{\nabla}\cdot\vec{\psi}/C_o, \tag{9}$$

where ∇^2 is the usual Laplace operator and $\vec{\nabla}$ is the usual divergence.

Differentiating (8a) with respect to y and (8b) with respect to x and subtracting and continuing this process we obtain the three equations;

$$\nabla^2(\mathrm{curl}\,\delta\vec{s}) + h^2(\mathrm{curl}\,\delta\vec{s}) = -\frac{1}{\mu_o}\mathrm{curl}\left[\delta\rho\,\omega^2\vec{s}_o + \vec{F}\right]$$

$$= -\mathrm{curl}\,\vec{\psi}/\mu_o . \qquad (10)$$

The solution to (9) now represents the dilational component of the scattered wave which is produced when the incident wave passes through the inhomogeneity, and the solution to (10) represents the shear component to the scattered, wave. Here, we shall consider two sets of initial conditions, the first is a plane dilational wave propagated in the negative x direction. That is,

$$\vec{s}_o = u_o\underline{i} = A\exp(ikx)\,\underline{i} . \qquad (11a)$$

The second is the case in which we have a plane shear wave propagating in the negative x direction polarized in the y direction. That is,

$$\vec{s}_o = v_o\underline{j} = A\exp(ihx)\,\underline{j} . \qquad (11b)$$

7.3 SOLUTIONS FOR THE SCATTERED WAVE.

The particular solution to (9) which represents a spherical outgoing wave is given by

$$\vec{\nabla}\cdot\vec{\delta s} = +\frac{1}{4\pi C_o}\int \frac{e^{-ikr}}{r}\ \vec{\nabla}_c\cdot\vec{\psi}(x_o,y_o,z_o)\,d\tau_o\ , \qquad (12)$$

in which

$$r^2 = (x - x_o)^2 + (y - y_o)^2 + (z - z_o)^2, \qquad (13a)$$

$$\vec{\nabla}_c = \frac{\partial}{\partial x_o}\,\underline{i} + \frac{\partial}{\partial y_o}\,\underline{j} + \frac{\partial}{\partial z_o}\,\underline{k}\ , \qquad (13b)$$

$$d\tau_o = dx_o dy_o dz_o\ , \qquad (13c)$$

and the integration is over the whole volume of the solid. In this case the integration goes only over the volume of the inhomogeneity, since the δ's and their derivatives are zero everywhere except in the inhomogeneity and on its surface.*

Integrating (12) once by parts and applying the above boundary conditions, that the δ's and their derivatives are zero outside the inhomogeneity, and further using the fact that

$$\frac{\partial}{\partial x_o}\left(\frac{e^{-ikr}}{r}\right) = -\frac{\partial}{\partial x}\left(\frac{e^{-ikr}}{r}\right),$$

we obtain for (12),

$$\vec{\nabla}\cdot\vec{\delta s} = \vec{\nabla}\cdot\left[\frac{1}{4\pi C_o}\int \frac{e^{-ikr}}{r}\ \vec{\psi}(x_o,y_o,z_o)\ d\tau_o\right]. \qquad (14a)$$

* The integration of (12) follows that given by Rayleigh, Theory of Sound, (Dover Publications, New York, 1945), Vol. 2, p. 149.

In exactly the same way, the solution to (16) is given by

$$\text{curl}\vec{\delta s} = \text{curl}\left[\frac{1}{4\pi\mu_c}\int \frac{e^{-ikr}}{r}\vec{\psi}(x_o,y_o,z_o)d\tau_o\right]. \qquad (14b)$$

In (14a) and (14b) we have integrals of the type

$$J_1 = \int \frac{e^{-ikr}}{r}\omega^2\vec{s}_o(x_o,y_o,z_o)\,\delta\rho\,(x_o,y_o,z_o)d\tau_o \qquad (15a)$$

and $$J_2 = \int \frac{e^{-ikr}}{r}\frac{\partial}{\partial x_o}\left(\delta C_{ij}\,(x_o,y_o,z_o)S_j^o\,(x_o,y_o,z_o)\right)d\tau_o \qquad (15b)$$

to evaluate. Integrating J_2 once by parts and applying the boundary conditions above, then it reduces to an integral of the type J_1. That is, it becomes

$$J_2 = \frac{\partial}{\partial x}\int \frac{e^{-ikr}}{r}\,\delta C_{ij}(x_o,y_o,z_o)S_j^o(x_o,y_o,z_o)d\tau_o\,. \qquad (15c)$$

Remembering that the volume of integration is small compared to a wavelength of the ultrasonic wave, then $\frac{e^{-ikr}}{r}\,x\,S_j^o(x_o,y_o\,z_o)$ will be a slowly varying function over the region of integration and hence we may replace it by the value at the center of the inhomogeneity. In particular, if we take the origin of the coordinate system at the center of the inhomogeneity, then we may set

$$x_o = y_o = z_o = 0\,, \qquad (16)$$

and hence

$$J_1 = \frac{e^{-ikr}}{r}\omega^2\vec{s}_o(0,0,0)\int \delta\rho\, d\tau_o$$

$$= \frac{e^{-ikr}}{r}\omega^2\vec{s}_o(0,0,0)\,\Delta\rho\, T\,, \qquad (17a)$$

where now r is given by (13a) and (16), T is the volume of the inhomogeneity and $\Delta\rho$ is the average value of $\delta\rho$ over the inhomogeneity. In solids, it turns out that ρ can be considered to be constant over the volume of the inhomogeneity and hence $\delta\rho = \Delta\rho$ if we neglect boundary effects, which, in fact, will be negligible.

In the same manner,

$$J_2 = \frac{\partial}{\partial x}\left[\frac{e^{-ikr}}{r} S_j^{\circ}(0,0,0) \int \delta C_{ij}(x_o,y_o,z_o)d\tau_o\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{e^{-ikr}}{r} S_j^{\circ}(0,0,0)T\Delta C_{ij}\right], \qquad (17b)$$

where ΔC_{ij} is the average of δC_{ij} over the volume of the inhomogeneity. Our model gives δC_{ij} to be constant over the volume of the inhomogeneity except at its surface. For all practical purposes, these surface effects are negligible in ΔC_{ij} and we may set $\Delta C_{ij} = \delta C_{ij}$. This is done throughout the rest of our calculations. Further, we note that $\vec{r}$, given by (13a) and (16), is the outward vector from the center of the inhomogeneity along which the scattered wave propagates.

Using (4), (8d), (9), (14a), (14b), (17a), and (17b) together with each of (11a) and (11b) we have the solutions for the incident dilational waves and the incident shear waves:

(i) The incident plane dilational wave prepresented by (11a);

$$\vec{\nabla}\cdot\delta\vec{s} = \vec{\nabla}\cdot\frac{AT}{4\pi C_o}\left[\omega^2\Delta\rho R\underline{i} + ik\vec{H}_1\right], \qquad (18a)$$

and

$$\text{curl}\,\delta\vec{s} = \text{curl}\,\frac{AT}{4\pi\mu_o}\left[\omega^2\Delta\rho Q\underline{i} + ik\vec{\mathcal{H}}_1\right], \qquad (18b)$$

where

$$R = \frac{e^{-ikr}}{r}, \quad Q = \frac{e^{-ihr}}{r}, \tag{18c}$$

and $\vec{H}_1 = \left(\Delta C_{11} \frac{\partial R}{\partial x} + \Delta C_{16} \frac{\partial R}{\partial y} + \Delta C_{15} \frac{\partial R}{\partial z} \right) \underline{i} + \left(\Delta C_{16} \frac{\partial R}{\partial x} + \right.$

$$\left. \Delta C_{12} \frac{\partial R}{\partial y} + \Delta C_{14} \frac{\partial R}{\partial z} \right) \underline{j} + \left(\Delta C_{15} \frac{\partial R}{\partial x} + \Delta C_{14} \frac{\partial R}{\partial y} + \Delta C_{13} \frac{\partial R}{\partial z} \right) \underline{k}, \tag{18d}$$

and $\vec{\mathcal{H}}_1$ is simply $\vec{H}_1$ with R replaced by Q.

(ii) The incident plane shear wave represented by (11b);

$$\vec{\nabla} \cdot \delta\vec{s} = \vec{\nabla} \cdot \frac{AT}{4\pi C_0} \left[\omega^2 \Delta\rho R \underline{j} + ih\vec{H}_6 \right], \tag{19a}$$

and $\text{curl}\,\delta\vec{s} = \text{curl}\, \frac{AT}{4\pi\mu_0} \left[\omega^2 \Delta\rho\, Q\, \underline{j} + ih\vec{\mathcal{H}}_6 \right]$, (19b)

where the quantities are all as above except that H_6 and $\mathcal{H}_6$ are H_1 and $\vec{\mathcal{H}}_1$ with ΔC_{1j} replaced by ΔC_{6j}.

At distances far from the inhomogeneity we can neglect $1/r$ compared to unity and the operations

$$\frac{\partial}{\partial x}\left(\frac{e^{-ikr}}{r}\right) = -ikx\frac{e^{-ikr}}{r^2} = -ikl\frac{e^{-ikr}}{r}, \tag{20}$$

etc. We define the direction cosines of the vector $\vec{r}$ as

$$L = \frac{x}{r}, \quad M = \frac{y}{r}, \quad N = \frac{z}{r}.$$

That is L, M, and N are just the direction cosines of the outward going vector from the inhomogeneity.

The solutions (18a) and (18b) can now be rewritten as

$$\vec{\nabla}\cdot\vec{\delta s} = \vec{\nabla}\cdot\frac{ATR}{4\pi C_o}\left[\omega^2\Delta\rho\,\underline{i} + k^2\vec{H}_1^o\right], \tag{21a}$$

$$\text{and}\quad \text{curl}\,\vec{\delta s} = \text{curl}\,\frac{ATQ}{4\pi\mu_o}\left[\omega^2\Delta\rho\,\underline{i} + khH_1^o\right], \tag{21b}$$

where

$$\vec{H}_1^o = (L\Delta C_{11} + M\Delta C_{16} + N\Delta C_{15})\underline{i} + (L\Delta C_{16} + M\Delta C_{12} + N\Delta C_{14})\underline{j} + (L\Delta C_{15} + M\Delta C_{14} + N\Delta C_{13})\underline{k}. \tag{21c}$$

Similarly, the equations (19a) and 19b) can now be rewritten as

$$\vec{\nabla}\cdot\vec{\delta s} = \vec{\nabla}\cdot\frac{ATR}{4\pi C_o}\left[\omega^2\Delta\rho\,\underline{i} + khH_6^o\right], \tag{22a}$$

and

$$\text{curl}\,\vec{\delta s} = \text{curl}\,\frac{ATQ}{4\pi\mu_o}\left[\omega^2\Delta\rho\,\underline{i} + h^2\vec{H}_6^o\right], \tag{22b}$$

where $\vec{H}_1^o$ and H_6^o are related in the same way as $\vec{H}_1$ is to $\vec{H}_6$.

Finally, to the first order, given by (20), the solutions of $\vec{\delta s}$ at distances far from the inhomogeneity can be put in the form

$$\vec{\delta s} = \vec{S}_\rho + \vec{S}_s. \tag{23}$$

The solutions for the incident dilational wave are obtained from (21a) and 21b) resulting in

$$\vec{S}_\rho = \frac{ATR}{4\pi C_o}\left[(\omega^2\Delta\rho\,\underline{i} + k^2\vec{H}_1^o)\cdot\vec{n}\right]\vec{n}, \tag{24a}$$

and

$$\vec{S}_s = -\frac{ATQ}{4\pi\mu_o}\left[(\omega^2\Delta\rho\,\underline{i} + khH_1^o)\times\vec{n}\right]\times\vec{n}, \tag{24b}$$

where $\vec{n}$ is the outward going unit vector,

$$\vec{n} = L\underline{i} + M\underline{j} + N\underline{k} \quad . \tag{25}$$

The solutions for the incident shear wave are obtained from (22a) and (22b) resulting in

$$\vec{S}_\ell = \frac{ATR}{4\pi C_o} \left[\left(\omega^2 \Delta\rho \underline{j} + kh\vec{H}_\epsilon^o \right) \cdot \vec{n} \right] \vec{n} \quad , \tag{26a}$$

and

$$\vec{S}_s = - \frac{ATQ}{4\pi\mu_o} \left[\left(\omega^2 \Delta\rho \underline{j} + h^2 \vec{H}_\epsilon^o \right) \times \vec{n} \right] \times \vec{n} \quad . \tag{26b}$$

In the above expressions we see that $\vec{S}_\ell$ represents a spherical outgoing dilational wave propagating in the direction of the radius vector at a velocity V_ℓ and with displacement along the radius vector. Similarly, $\vec{S}_s$ represents a spherical outgoing shear wave, propagating with velocity V_s along the radius vector. Its displacement is perpendicular to the radius vector and its polarization is a function of the angle of scattering.

8. THE ATTENUATION COEFFICIENT.

8.1 THE SCATTERING CROSS-SECTION FOR A SINGLE INHOMOGENEITY.

We define the scattering cross-section, $\sigma(\theta,\varphi)$, as the energy scattered per unit time into a unit solid angle, centred at the scattering center, divided by the incident energy passing unit area perpendicular to the direction of propagation of the sound wave per unit time. First, the total energy, due to the incident wave, (11a), in an element of volume dV is

$$dE_\ell = IdV = \frac{1}{2}|i\omega A|^2 \rho_o \, dV \, , \tag{27}$$

where I is the intensity of the sound wave, that is,

$$I = \frac{1}{2}\omega^2 A^2 \rho_o \, . \tag{28}$$

If we have a volume V with unit area perpendicular to the direction of propagation of the sound wave and with length V_ℓ, then

$$W_\ell = I.1.V_\ell = \frac{1}{2}\rho_o \omega^2 A^2 V_\ell \tag{29}$$

represents the incident energy passing unit area per unit time.

Similarly, the energy passing unit area, perpendicular to the direction of propagation of the sound wave, per unit time due to the incident sound wave, (11b), is

$$W_s = \frac{1}{2}\rho_o \omega^2 A^2 V_s \, . \tag{30}$$

Also, the energy passing an element of area, dA, on the surface of a sphere with the inhomogeneity at its center due to the spherical outgoing scattered dilational and shear waves is,

$$\delta W = \frac{1}{2}\rho_o \omega^2 \left(S_\ell^2 V_\ell + S_s^2 V_s \right) dA \,, \tag{31}$$

where now S_ℓ and S_s are the maximum values of $\vec{S}_\ell$ and $\vec{S}_s$, respectively, and $dA = r^2 \sin\theta d\theta d\phi$.

Hence, by definition,

$$\sigma_\ell(\theta,\phi)d\Omega = \frac{1}{A^2 V_\ell} \left(S_\ell^2 V_\ell + S_s^2 V_s \right) dA \,, \tag{32}$$

and

$$\sigma_s(\theta,\phi)d\Omega = \frac{1}{A^2 V_s} \left(S_\ell^2 V_\ell + S_s^2 V_s \right) dA \,, \tag{33}$$

where $d\Omega = \sin\theta d\theta d\phi$ and we use (24a) and (24b) for $\vec{S}_\ell$ and $\vec{S}_s$ in (32), and (26a) and (26b) for $\vec{S}_\ell$ and $\vec{S}_s$ in (33).

The total cross-section, σ , is now obtained by integrating $\sigma(\theta,\phi)$ over all angles of scattering. This integration is quite straightforward and expressions (32) and (33) need only be written out fully. The result for (32) is

$$\sigma_\ell = \frac{T^2}{4\pi C_o^2} \Bigg[\frac{1}{3}\omega^4(\Delta\rho)^2 + \frac{1}{5}k^4 \Big[\Big\{ (\Delta C_{11})^2 + (\Delta C_{12})^2 + (\Delta C_{13})^2 \Big\} + \frac{4}{3}\Big\{ (\Delta C_{16})^2 + (\Delta C_{15})^2 + (\Delta C_{14})^2 \Big\} + \frac{2}{3}\Big\{ (\Delta C_{11})(\Delta C_{12}) + (\Delta C_{11})(\Delta C_{13}) + (\Delta C_{12})(\Delta C_{13}) \Big\} \Big] \Bigg] + \frac{V_s}{V_\ell} \frac{T^2}{4\pi\mu_o^2} \Bigg[\frac{2}{3}\omega^4(\Delta\rho)^2 + \frac{2k^2h^2}{15} \Big[\Big\{ (\Delta C_{11})^2 + (\Delta C_{12})^2 + (\Delta C_{13})^2 \Big\} + 3\Big\{ (\Delta C_{16})^2 + (\Delta C_{15})^2 + (\Delta C_{14})^2 \Big\} - \Big\{ (\Delta C_{11})(\Delta C_{12}) + (\Delta C_{11})(\Delta C_{13}) + (\Delta C_{12})(\Delta C_{13}) \Big\} \Big] \Bigg].$$

Rewriting, this expression can be put in the form

$$\sigma_\ell = \frac{T^2\omega^4}{4\pi C_o^2} \left[\left\{ \frac{1}{3}(\Delta\rho)^2 + \frac{k^4}{5\omega^4} B_1 \right\} + \frac{V_s C_o^2}{V_\ell \mu_o^2} \left\{ \frac{2}{3}(\Delta\rho)^2 + \frac{2k^2h^2}{5\omega^4} A_1 \right\} \right]$$

$$= \frac{4\pi^3 T^2 f^4}{V_\rho^4}\left[\frac{1}{3}\left(1 + 2\left(\frac{V_\rho}{V_s}\right)^3\right)\frac{(\Delta\rho)^2}{\rho_o^2} + \frac{1}{5C_o^2}\left(B_1 + 2\left(\frac{V_\rho}{V_s}\right)^5 A_1\right)\right]$$

$$= \frac{4\pi^3 T^2 f^4}{V_\rho^3 V_s}\frac{C_o^2}{\mu_o^2}\left[\frac{2\mu_o}{3C_o}\left(1 + \frac{1}{2}\left(\frac{V_s}{V_\rho}\right)^3\right)\frac{(\Delta\rho)^2}{\rho_o^2} + \frac{2}{5C_o^2}\left(A_1 + \frac{1}{2}\left(\frac{V_s}{V_\rho}\right)^5 B_1\right)\right]. \tag{34}$$

In the same way, we obtain

$$\sigma_s = \frac{T^2}{4\pi C_o^2}\frac{V_\rho}{V_s}\left[\frac{1}{3}\omega^4(\Delta\rho)^2 + \frac{k^2h^2}{5}B_6\right] + \frac{T^2}{4\pi\mu_o^2}\left[\frac{2}{3}\omega^4(\Delta\rho)^2 + \frac{2h^4}{5}A_6\right]$$

$$= \frac{4\pi^3 T^2 f^4}{V_s^4}\frac{C_o^2}{\mu_o^2}\left[\frac{2\mu_o^2}{3C_o^2}\left(1 + \frac{1}{2}\left(\frac{V_s}{V_\rho}\right)^3\right)\frac{(\Delta\rho)^2}{\rho_o^2} + \frac{2}{5C_o^2}\left(A_6 + \frac{1}{2}\left(\frac{V_s}{V_\rho}\right)^5 B_6\right)\right]. \tag{35}$$

In (34) and (35),

$$A_i = \frac{1}{3}\left((\Delta C_{i1})^2 + (\Delta C_{i2})^2 + (\Delta C_{i3})^2\right) + \left((\Delta C_{i6})^2 + (\Delta C_{i5})^2 + (\Delta C_{i4})^2\right) - \frac{1}{3}\left((\Delta C_{i1})(\Delta C_{i2}) + (\Delta C_{i1})(\Delta C_{i3}) + (\Delta C_{i2})(\Delta C_{i3})\right), \tag{36}$$

and

$$B_i = \left((\Delta C_{i1})^2 + (\Delta C_{i2})^2 + (\Delta C_{i3})^2\right) + \frac{4}{3}\left((\Delta C_{i6})^2 + (\Delta C_{i5})^2 + (\Delta C_{i4})^2\right) + \frac{2}{3}\left((\Delta C_{i1})(\Delta C_{i2}) + (\Delta C_{i1})(\Delta C_{i3}) + (\Delta C_{i2})(\Delta C_{i3})\right), \tag{37}$$

We note that (34), (35), (36), and (37) apply to all crystal symmetries. However, we have limited the following calculations to crystals of orthorhombic symmetry.

8.2 THE ATTENUATION COEFFICIENT FOR A POLYCRYSTAL.

If our only scattering mechanism is that of varying orientation of the grains, then we may calculate a value for the attenuation coefficient from the preceding model. To do this we need to make the following assumptions: (1) All the inhomogeneities are very nearly the same size, T. (2) Each grain scatters independently of the others (assumption of random orientations of the grains), and (3) The variations in the density from grain to grain are uncorrelated. The corresponding assumption about the elastic constants is contained in (2). These assumptions enable us to sum over the scattered energies from the individual grains.

Remembering the definition of the attenuation coefficient, it is the reciprocal of the distance in which the intensity of the sound wave drops to 1/e of its value. Hence it is defined by

$$\alpha\, dx = -\frac{dI}{I}, \tag{38}$$

or

$$I = I_o \exp(-\alpha x),$$

where I_o is the intensity at the reference plane x = 0, I is the intensity at x, and dI is the change in the intensity in the distance dx. From (29) we see that dI/I is equal to the energy scattered out of the volume Sdx per unit time divided by the energy passing the area S per unit time, where S is an area taken perpendicular to the direction of propagation of the sound wave. Hence,

$$\frac{dI}{I} = -\frac{\sum_{\substack{\text{all grains} \\ \text{in } Sdx}} \left\{\frac{\sigma}{S}\right\}}{} = -\frac{Sdx}{T}\,\frac{\langle\sigma\rangle}{S}$$

$$= -\frac{1}{T}\langle\sigma\rangle dx \quad , \tag{39}$$

where T is the volume of a single grain, and $\langle\sigma\rangle$ is the average value of σ over all grain orientations.

Comparing (38) and (39) then

$$\alpha = \frac{\langle\sigma\rangle}{T} \ .$$

From (34) and (35) we have

$$\alpha_\ell = \frac{4\pi^3 T f^4}{V_\ell^3 V_s} \frac{C_o^2}{\mu_o^2} \left[\frac{2\mu_o}{3C_o} \left\{ 1 + \frac{1}{2}\left(\frac{V_s}{V_\ell}\right)^3 \right\} \frac{\langle(\Delta\rho)^2\rangle}{\rho_o^2} \right.$$

$$\left. + \frac{2}{5C_o^2} \left\{ \langle A_1 \rangle + \frac{1}{2}\left(\frac{V_s}{V_\ell}\right)^5 \langle B_1 \rangle \right\} \right] , \tag{40}$$

and

$$\alpha_s = \frac{4\pi^3 T f^4}{V_s^4} \frac{C_o^2}{\mu_o^2} \left[\frac{2\mu_o^2}{3C_o^2} \left(1 + \frac{1}{2}\left(\frac{V_s}{V_\ell}\right)^3 \right) \frac{\langle(\Delta\rho)^2\rangle}{\rho_o^2} \right.$$

$$\left. + \frac{2}{5C_o^2} \left\{ \langle A_6 \rangle + \frac{1}{2}\left(\frac{V_s}{V_\ell}\right)^5 \langle B_6 \rangle \right\} \right] , \tag{41}$$

The averaging of A_i and B_i over all grain orientations is carried out in Appendix I and the results are stated below.

$$\langle A_1 \rangle = \langle B_6 \rangle = \frac{2P^2}{225} + \frac{1}{135}(24a + 7b + 13c + d),$$

$$\langle B_1 \rangle = \frac{8P^2}{675} + \frac{8}{135} (4a + 2b + 3c + d) ,$$

$$\langle A_6 \rangle = \frac{1P^2}{150} + \frac{1}{90} (12a + b + 4c - 2d) , \tag{42}$$

where

$$P = (C_{11} + C_{22} + C_{33}) - (C_{12} + C_{13} + C_{23})$$

$$- 2(C_{44} + C_{55} + C_{66}) ,$$

$$a = (C_{44} + C_{55} + C_{66})^2 - 3(C_{44}C_{55} + C_{44}C_{66} + C_{55}C_{66}) ,$$

$$b = (C_{11} + C_{22} + C_{33})^2 - 3(C_{11}C_{22} + C_{11}C_{33} + C_{22}C_{33}) ,$$

$$c = (C_{12} + C_{13} + C_{23})^2 - 3(C_{12}C_{13} + C_{12}C_{23} + C_{13}C_{23}) ,$$

$$d = C_{11}(C_{12} + C_{13} - 2C_{23}) + C_{22}(C_{12} + C_{23} - 2C_{13}) + C_{33}(C_{13} + C_{23} - 2C_{12}) . \qquad (43)$$

Expressions for C_o, μ_o, $\langle A_i \rangle$, and $\langle B_i \rangle$ appropriate for crystals of higher symmetry than orthorhombic may now be obtained as special cases of (42) and (43). For example, we have to put:

(i) For isotropic crystals;

$$C_{11} = C_{22} = C_{33} = C = \lambda + 2\mu ,$$

$$C_{44} = C_{55} = C_{66} = \mu ,$$

$$C_{12} = C_{13} = C_{23} = \lambda . \qquad (44a)$$

(ii) For cubic crystals:

$$C_{11} = C_{22} = C_{33} ,$$

$$C_{44} = C_{55} = C_{66} ,$$

$$C_{12} = C_{13} = C_{23} . \qquad (44b)$$

(iii) For hexagonal crystals;

$$C_{11} = C_{22} , \quad C_{13} = C_{23} , \quad C_{44} = C_{55} ,$$

$$C_{66} = \frac{1}{2}(C_{11} - C_{12}) , \quad C_{33} , \quad C_{12} . \qquad (44c)$$

α_ℓ , (40), and α_s , (41), represent the intensity attenuation

coefficients. Since the intensity is proportional to the square of the amplitude, the amplitude attenuation coefficient is just one-half that for the intensity. Hence

$$D_{2\ell} = \frac{\alpha_\ell}{2f^4} \quad \text{(nepers/cm./cycle}^4\text{)} , \tag{45a}$$

and

$$D_{2s} = \frac{\alpha_s}{2f^4} \quad \text{(nepers/cm./cycle}^4\text{)} . \tag{45b}$$

9. DISCUSSION

The terms in the attenuation coefficients, (40) and (41), arising from the variations in the density are exactly those obtained by Bhatia, (1959) [17]. He has discussed these terms fully. Both the variation in density from grain to grain and the variation in density in an individual grain due to thermal agitation are negligible in solids.

Neglecting these density terms, then*

$$D_{2\ell} = \frac{4\pi^3 T}{5V_\ell^3 V_s \mu_0^2}\left[\langle A_1\rangle + \frac{1}{2}\left(\frac{V_s}{V_\ell}\right)^5 \langle B_1\rangle\right], \tag{46a}$$

and

$$D_{2s} = \frac{4\pi^3 T}{5V_s^4 \mu_0^2}\left[\langle A_6\rangle + \frac{1}{2}\left(\frac{V_s}{V_\ell}\right)^5 \langle B_6\rangle\right]. \tag{46b}$$

The first term in the square brackets arises from the scattered shear wave and the second term arises from the scattered dilational wave. Since $V_s/V_\ell \sim 1/2$ for most solids and $\langle A_i\rangle \sim \langle B_i\rangle$, generally, the second term is usually only a few percent of the first. This accounts for the discrepency in Mason and McSkimin's, [15], [16], theoretical value for the dilational waves as they take only the second term into account. (Of course, they realized the limitations of their calculations.)

In crystals of cubic symmetry,

$$\langle A_1\rangle = \langle B_6\rangle = \frac{2p^2}{25}, \quad \langle A_6\rangle = \frac{3p^2}{50}, \quad \langle B_1\rangle = \frac{8p^2}{75},$$

where $p = C_{11} - C_{12} - 2C_{44}$. Hence,

---

*We note that our D's correspond to the B's appearing in [16] and [17].

$$D_{2S} = \frac{3}{4}\left(\frac{V}{V_S}\ell\right)^3 D_{2\ell} = \frac{6\pi^3 T\, p^2}{125 V_S^4 \mu_o^2}\left[\,1 + \frac{2}{3}\left(\frac{V_S}{V_\ell}\right)^5\,\right]. \tag{47}$$

Since $V_S/V_\ell \sim 1/2$, then

$$\frac{D_{2S}}{D_{2\ell}} \sim 6\ . \tag{48}$$

(48) is just the same ratio as that obtained by Bhatia, [17]. However, we see now that (47) is valid only for crystals of cubic symmetry. For crystals of other symmetry $D_{2S}/D_{2\ell}$ depends on the values of the elastic constants themselves through the $\langle A_i \rangle$'s and the $\langle B_i \rangle$'s. The agreement between the theoretical and the experimental values of $D_{2S}/D_{2\ell}$, evaluated in Table 2, is reasonable for aluminum but is rather poor for the magnesium sample.

We mention that the measurements have been made on two aluminum samples having different average grain diameters. The dependence of the attenuation coefficient on grain size is in accordance with (46) to within five percent. The dependence of attenuation on grain size is the same in all three theories; Mason and McSkimin's, [16], Bhatia's, [17], and ours. We note that $D_{2\ell}$ and D_{2S} depend quite critically on the average grain diameter, as the third power.

Finally, Table 2 contains the quantitative values for $D_{2\ell}/T$ and D_{2S}/T for the aluminum and the magnesium samples. As we have already mentioned, Mason and McSkimin obtain values much too small for $D_{2\ell}$ and close to the experimental value for D_{2S}. Bhatia, on the other hand, now obtains values in quite good agreement with the experimental values for both $D_{2\ell}$ and D_{2S}. Our values are several times too large for both $D_{2\ell}$ and D_{2S}, thus making the agreement poorer with experiment.

In the case of magnesium, the discrepency between experiment and the various theories is much larger.* It appears to us to be somewhat premature to comment upon these discrepancies in view of the following. First, there are very few experimental data available - only one set of data for magnesium. Secondly, as we have seen, the attenuation depends sensitively on the grain size which is not always easy to measure accurately. There are also the various simplifying assumptions that we have made in our calculations, in particular, the random orientations of the grains and small elastic anisotropy, which may not be completely applicable to the experimental data available. Hence, more experimental measurements would be desirable.

---

*Mason and McSkimin, (1947) [15], quote a theoretical value for D_{2S} for magnesium which is in very good agreement with their experimental value. However, they obtained this agreement by using a value for $\frac{\langle \Delta C_{44}^2 \rangle}{\mu_0^2} \simeq 4 \times 10^{-5}$, where, in fact, this ratio is about 4×10^{-3}.

	Mason & McSkimin	Bhatia	Ours	Observed
Aluminum				
$D_{2\ell}/T \times 10^{24}$	0.096	1.29	4.51	0.61
$D_{2s}/T \times 10^{24}$	13.0	10.6	38.2	8.2
$D_{2s}/D_{2\ell}$	135	8.3	8.4	13.5
Magnesium				
$D_{2\ell}/T \times 10^{24}$	0.17	1.06	9.0	...
$D_{2s}/T \times 10^{24}$	20.2	12.5	42.8	0.23
$D_{2s}/D_{2\ell}$	119	11.8	4.7	1

TABLE II

Values of $D_{2\ell}/T$, D_{2s}/T, and $D_{2s}/D_{2\ell}$ for aluminum and magnesium.

APPENDIX I. METHOD OF AVERAGING OVER RANDOM GRAIN ORIENTATIONS.

The elastic constants transform, under coordinate transformations, as the components of a fourth rank tensor and for the purposes of transformations it is convenient to relabel them. Hence, in this section we write;

$$C_{11} = C_{1111}\ ,\quad C_{12} = C_{1122}\ ,\quad C_{13} = C_{1133}\ ,$$
$$C_{14} = C_{1123}\ ,\quad C_{15} = C_{1113}\ ,\quad C_{16} = C_{1112}\ ,$$
$$C_{22} = C_{2222}\ ,\quad C_{23} = C_{2233}\ ,\quad C_{24} = C_{2223}\ ,$$
$$C_{25} = C_{2213}\ ,\quad C_{26} = C_{2212}\ ,\quad C_{33} = C_{3333}\ , \qquad \text{A.1}$$
$$C_{34} = C_{3323}\ ,\quad C_{35} = C_{3313}\ ,\quad C_{36} = C_{3312}\ ,$$
$$C_{44} = C_{2323}\ ,\quad C_{45} = C_{2313}\ ,\quad C_{46} = C_{2312}\ ,$$
$$C_{55} = C_{1313}\ ,\quad C_{56} = C_{1312}\ ,\quad C_{66} = C_{1212}\ .$$

Further, we note that $C_{ij} = C_{ji}$ and that

$$C_{ijkl} = C_{ijlk} = C_{jikl} = C_{jilk}\ , \qquad \text{A.2}$$

The transformed elastic constants are given by

$$C'_{ijkl} = a_{im}a_{jn}a_{ko}a_{lp}C_{mnop}\ , \qquad \text{A.3}$$

where again the repeated indices mean summing over that index and where the a_{im}'s define the rotation of the crystal from an orientation in which its crystallographic axes coincide with the laboratory coordinate system to its actual orientation in the solid. This rotation may be defined in terms of Euler angles α, β, and γ and the rotation is carried out in the three stages: (1) rotate about the initial z-axis by

the angle α in a clockwise direction, (2) rotate about the new x-axis by the angle β in the clockwise direction and, (3) rotate about the new z-axis by the angle γ. These rotations are illustrated in figure 6. This transformation is given by;

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad \text{A.4a}$$

with

$$A = \begin{pmatrix} \cos\alpha\cos\gamma - \sin\alpha\cos\beta\sin\gamma & -\sin\alpha\cos\gamma - \cos\alpha\cos\beta\sin\gamma & \sin\beta\sin\gamma \\ \cos\alpha\sin\gamma + \sin\alpha\cos\beta\cos\gamma & -\sin\alpha\sin\gamma + \cos\alpha\cos\beta\cos\gamma & -\sin\beta\cos\gamma \\ \sin\alpha\sin\beta & \cos\alpha\sin\beta & \cos\beta \end{pmatrix}, \qquad \text{A.4b}$$

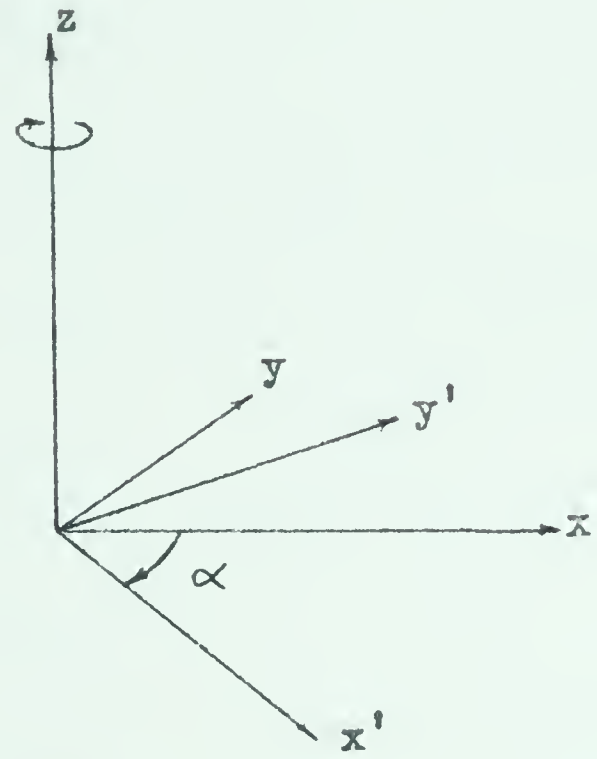
where $0 \leq \alpha \leq 2\pi$, $0 \leq \beta \leq \pi$, and $0 \leq \gamma \leq 2\pi$. A.4c

To average the quantities in A.3 over all angles of rotation we need to write

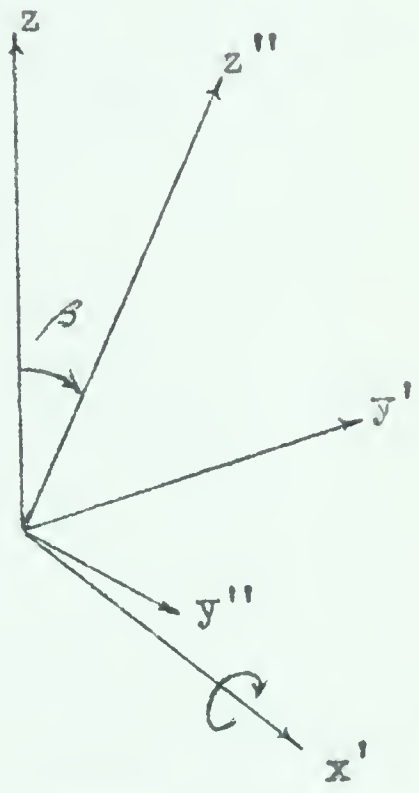
$$\langle C'_{ijkl} \rangle = \frac{1}{8\pi^3} \int C'_{ijkl} \sin\beta \, d\beta \, d\alpha \, d\gamma \, , \qquad \text{A.5}$$

where the integrations go over the whole range of the angles involved and where the element $\sin\beta \, d\beta \, d\alpha$ is the solid angle produced by the first two rotations and the $d\gamma$ is the element of angle over the second rotation.

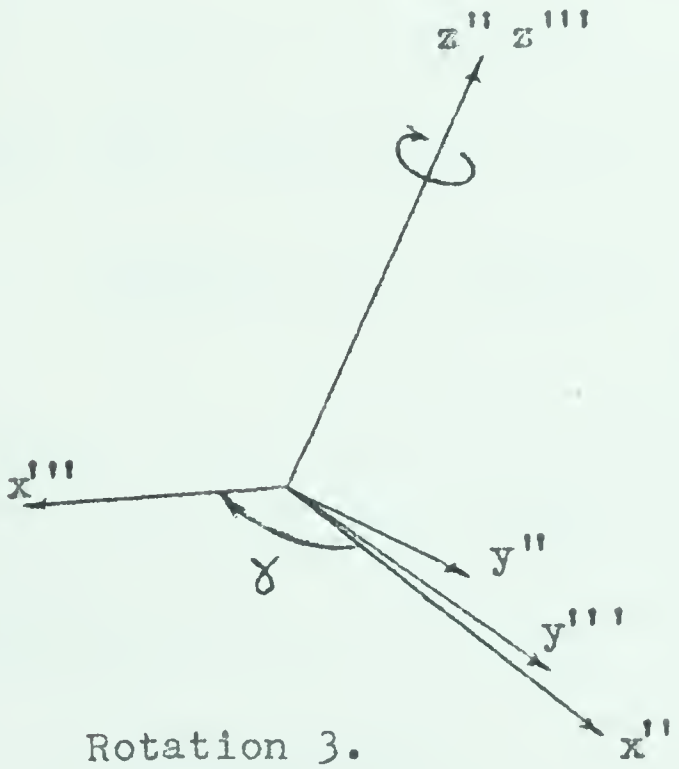
In the case of the average over the product of four elements from the rotation matrix, A, we have the following cases with their values;



Rotation 1.



Rotation 2.



Rotation 3.

Figure 6.

The Euler Angles, α, β and γ.

(a) All four a's are the same;

$$\langle a_{im}^{4} \rangle = 1/5 . \qquad \text{A.6a}$$

(b) Two a's are the same and two are the same, excluding (a). There are now two possibilities;

(i) Both from the same row or column;

$$\langle a_{im}^{2} a_{in}^{2} \rangle = \langle a_{im}^{2} a_{jm}^{2} \rangle = 1/15 . \qquad \text{A.6bi}$$

(ii) Both from different rows and columns;

$$\langle a_{im}^{2} a_{jn}^{2} \rangle = 2/15 , \qquad \text{A.6bii}$$

(c) Three of the a's are the same and one is different. In all such cases all the averages are zero.

$$\langle a_{im}^{3} a_{jn} \rangle = 0 . \qquad \text{A.6c}$$

(d) Three different a's with one squared. All averages are zero.

$$\langle a_{im}^{2} a_{in} a_{io} \rangle = \langle a_{im}^{2} a_{jm} a_{km} \rangle = \langle a_{im}^{2} a_{jn} a_{ko} \rangle = 0 . \qquad \text{A.6d}$$

(e) All four a's are different then we have only one case of a non-zero average and that is when the a's in the product form a minor of some element in the original transformation matrix, A, and here

$$\langle a_{in} a_{im} a_{jn} a_{jm} \rangle = -1/30 . \qquad \text{A.6e}$$

To obtain the desired average values it is necessary to expand A.3 into its full form and substitute the above averages. When this is done, we have

$$C_o = \langle C'_{1111} \rangle = \langle C'_{2222} \rangle = \langle C'_{3333} \rangle$$

$$= \frac{1}{3}(C_{1111} + C_{2222} + C_{3333}) - \frac{2P}{15}$$

$$= \frac{1}{3}(C_{11} + C_{22} + C_{33}) - \frac{2P}{15} , \qquad \text{A.7a}$$

where

$$P = (C_{1111} + C_{2222} + C_{3333}) - (C_{1122} + C_{1133} + C_{2233})$$

$$- 2(C_{1212} + C_{1313} + C_{2323})$$

$$= (C_{11} + C_{22} + C_{33}) - (C_{12} + C_{13} + C_{23})$$

$$- 2(C_{44} + C_{55} + C_{66}) , \qquad \text{A.7b}$$

and

$$\mu_o = \langle C'_{1212} \rangle = \langle C'_{1313} \rangle = \langle C'_{2323} \rangle$$

$$= \frac{1}{3}(C_{1212} + C_{1313} + C_{2323}) + \frac{1P}{15}$$

$$= \frac{1}{3}(C_{44} + C_{55} + C_{66}) + \frac{1P}{15} , \qquad \text{A.7c}$$

and

$$\lambda_o = \langle C'_{1122} \rangle = \langle C'_{1133} \rangle = \langle C'_{2233} \rangle$$

$$= \frac{1}{3}(C_{1122} + C_{1133} + C_{2233}) + \frac{1P}{15}$$

$$= \frac{1}{3}(C_{12} + C_{13} + C_{23}) + \frac{1P}{15} . \qquad \text{A.7d}$$

These relations are valid for crystals with 21 elastic constants and can be reduced to apply to crystals of higher symmetry by use of equations (44a) to (44c) in section 8.2. We note that $C_o = \lambda_o + 2\mu_o$ as is the case for isotropic crystals and that only nine of the original 21 elastic constants appear at this point. These expressions reduce

to those calculated by Mason and McSkimin for crystals of cubic symmetry. All other averages turn out to be zero.

To evaluate the average values appropriate for section 8.2 we need to evaluate expressions of the type $\langle\langle(\Delta C_{im})\ (\Delta C_{in})\rangle\rangle$. In our model $\Delta C_{im} = \delta C_{im}$ and we have set

$$\delta C_{im} = C'_{im} - C^{o}_{im}$$

from (9). Hence,

$$(\Delta C_{im})\ (\Delta C_{in}) = (C'_{im} - C^{o}_{im})\ (C'_{in} - C^{o}_{in})$$

$$= C'_{im}C'_{in} - C^{o}_{im}C'_{in} - C'_{im}C^{o}_{in} + C^{o}_{im}C^{o}_{in},$$

and therefore,

$$\langle(\Delta C_{im})\ (\Delta C_{in})\rangle = \langle C'_{im}C'_{in}\rangle - C^{o}_{im}C^{o}_{in}, \qquad \text{A.8}$$

Thus we are left to evaluate the averages of expressions like $C'_{im}C'_{in}$ or in tensor notation,

$$C'_{ijkl}C'_{abcd} = a_{im}a_{jn}a_{ko}a_{lp}a_{ae}a_{bf}a_{cg}a_{dh}C_{mnop}C_{efgh}, \qquad \text{A.9}$$

where all the repeated indices are summed over. Thus we have to calculate the average values of all possible products of eight a's. All the possible values are tabulated. In the following repeated indices are not summed and $i \neq j \neq k \neq l$ and $m \neq n \neq o \neq p$.

(a) 8 alike;

$$\langle a^{8}_{im}\rangle = 1/9, \qquad \text{A.10a}$$

(b) 7 alike and 1 different;

All zero. A.10b

(c) 6 alike and 2 alike;

$$\langle a_{im}^{6} a_{in}^{2} \rangle = \langle a_{im}^{6} a_{jm}^{2} \rangle = 1/63 \ , \qquad \langle a_{im}^{6} a_{jn}^{2} \rangle = 4/63 \ . \qquad \text{A.10c}$$

(d) 6 alike and 2 different;

All zero . A.10d

(e) 5 alike and 3 alike;

All zero . A.10e

(f) 5 alike, 2 alike, and 1 different;

All zero . A.10f

(g) 5 alike and 3 different;

All zero, except

$$\langle a_{im}^{5} a_{in} a_{jm} a_{jn} \rangle = -\frac{1}{2} \cdot \frac{1}{63} \ . \qquad \text{A.10g}$$

(h) 4 alike and 4 alike;

$$\langle a_{im}^{4} a_{in}^{4} \rangle = \langle a_{im}^{4} a_{jm}^{4} \rangle = \frac{1}{105} \ , \qquad \langle a_{im}^{4} a_{jn}^{4} \rangle = \frac{2}{35} \ . \qquad \text{A.10h}$$

(i) 4 alike, 2 alike and 2 alike;

$$\langle a_{im}^{4} a_{jm}^{2} a_{km}^{2} \rangle = \langle a_{im}^{4} a_{in}^{2} a_{ip}^{2} \rangle = \frac{1}{315} \ ,$$

$$\langle a_{im}^{2} a_{jm}^{4} a_{jn}^{2} \rangle = \frac{2}{315} \ ,$$

$$\langle a_{im}^{4} a_{jm}^{2} a_{jn}^{2} \rangle = \frac{3}{315} \ ,$$

$$\langle a_{im}^{4} a_{jm}^{2} a_{kn}^{2} \rangle = \frac{4}{315} \ ,$$

$$\langle a_{im}^2 a_{jm}^2 a_{kn}^4 \rangle = \frac{6}{315} ,$$

$$\langle a_{im}^4 a_{jn}^2 a_{kp}^2 \rangle = \frac{17}{315} , \qquad \text{A.10i}$$

(j) 4 alike, 3 alike, and 1 different;

All zero . A.10j

(k) 4 alike, 2 alike and 2 different;

All zero . A.10k

(l) 4 alike and 4 differnnt;

All are zero except,

$$\langle a_{im} a_{in} a_{jm} a_{jn} a_{kn}^4 \rangle = -\frac{1}{2} \cdot \frac{1}{315} ,$$

$$\langle a_{im}^4 a_{jn} a_{kn} a_{jp} a_{kp} \rangle = -\frac{1}{2} \cdot \frac{11}{315} , \qquad \text{A.10l}$$

(m) 3 alike, 3 alike, and 2 alike;

All zero . A.10m

(n' 3 alike, 3 alike, and 2 different;

$$\langle a_{im}^3 a_{jm}^3 a_{in} a_{jn} \rangle = -\frac{1}{2} \cdot \frac{1}{105} ,$$

$$\langle a_{im}^3 a_{jm} a_{in} a_{jn}^3 \rangle = -\frac{1}{4} \cdot \frac{3}{105} , \qquad \text{A.10n}$$

and all others are zero.

(o) 3 alike, 2 alike, and 3 different;

$$\langle a_{im}^3 a_{in} a_{jm} a_{jn} a_{kn}^2 \rangle = -\frac{1}{4} \cdot \frac{3}{315} ,$$

$$\langle a_{kp}^2 a_{im}^3 a_{jm} a_{in} a_{jn} \rangle = -\frac{1}{4} \cdot \frac{13}{315} ,$$

$$\langle a_{im}^3 a_{in}^2 a_{ip} a_{jm} a_{jp} \rangle = -\frac{1}{2} \cdot \frac{1}{315} ,$$

$$\langle a^3_{im} a_{in} a^2_{jn} a_{km} a_{kn} \rangle = -\frac{1}{4}\cdot\frac{1}{315} . \qquad \text{A.10o}$$

(p) 3 alike, 2 alike, 2 alike, and 1 different;

All zero. A.10p

(p) 3 alike and 5 different;

$$\langle a_{im} a_{in} a_{jm} a_{jp} a_{kn} a^3_{kp} \rangle = \frac{1}{4}\cdot\frac{5}{315} , \qquad \text{A.10q}$$

Any one is arbitrarily cubed and all the others are zero.

(r) 2 alike, 2 alike, 2 alike, and 2 alike;

$$\langle a^2_{im} a^2_{jm} a^2_{jn} a^2_{jp} \rangle = \frac{1}{315} ,$$

$$\langle a^2_{im} a^2_{jm} a^2_{in} a^2_{jn} \rangle = \frac{1}{2}\cdot\frac{3}{315} ,$$

$$\langle a^2_{im} a^2_{in} a^2_{km} a^2_{kp} \rangle = \frac{1}{2}\cdot\frac{3}{315} ,$$

$$\langle a^2_{im} a^2_{in} a^2_{jp} a^2_{kp} \rangle = \frac{2}{315} ,$$

$$\langle a^2_{im} a^2_{in} a^2_{jn} a^2_{ko} \rangle = \frac{1}{2}\cdot\frac{7}{315} , \qquad \text{A.10r}$$

(s) 2 alike, 2 alike, 2 alike, and 2 different;

All zero. A.10s

(t) 2 alike, 2 alike, and 4 different;

$$\langle a_{im} a_{in} a_{jm} a_{jn} a^2_{kp} a^2_{jp} \rangle = -\frac{1}{315} ,$$

$$\langle a^2_{im} a_{in} a_{ip} a_{jn} a_{jp} a^2_{kp} \rangle = -\frac{1}{4}\cdot\frac{1}{315} . \qquad \text{A.10t}$$

All the others are zero.

(u) 2 alike and 6 different

$$\langle a_{im} a_{jm} a^2_{km} a_{kn} a_{kp} a_{in} a_{jp} \rangle = \frac{1}{2}\cdot\frac{1}{315} . \qquad \text{A.10u}$$

(v) 8 different;

All zero . A.10v

To evaluate the average of A.9 over all orientations one must write the expression out fully for each case of interest and use equations A.10a to A.10v to write down the results. For our purposes we need the following - which are written out for crystals of orthorhombic symmetry and where we have set

$$T_1 = C_{1111}^2 + C_{2222}^2 + C_{3333}^2 ,$$

$$T_2 = C_{2323}^2 + C_{1313}^2 + C_{1212}^2 ,$$

$$T_3 = C_{1122}^2 + C_{1133}^2 + C_{2233}^2 ,$$

$$T_4 = C_{1111}\,C_{2222} + C_{1111}\,C_{3333} + C_{2222}\,C_{3333} ,$$

$$T_5 = C_{1111}\,C_{1122} + C_{1111}\,C_{1133} + C_{2222}\,C_{1122} + C_{2222}\,C_{2233} + C_{3333}\,C_{1133} + C_{3333}\,C_{2233} ,$$

$$T_6 = C_{1111}\,C_{2233} + C_{2222}\,C_{1133} + C_{3333}\,C_{1122} ,$$

$$T_7 = C_{1111}\,C_{2323} + C_{2222}\,C_{1313} + C_{3333}\,C_{1212} ,$$

$$T_8 = C_{1111}\,C_{1313} + C_{1111}\,C_{1212} + C_{2222}\,C_{1212} + C_{2222}\,C_{2323} + C_{3333}\,C_{1313} + C_{3333}\,C_{2323} ,$$

$$T_9 = C_{1122}\,C_{1133} + C_{1122}\,C_{2233} + C_{1133}\,C_{2233} ,$$

$$T_{10} = C_{1122}\,C_{2323} + C_{1122}\,C_{1313} + C_{1133}\,C_{2323} + C_{1133}\,C_{1212} + C_{2233}\,C_{1212} + C_{2233}\,C_{1313} ,$$

$$T_{11} = C_{1122}\,C_{1212} + C_{1133}\,C_{1313} + C_{2233}\,C_{2323} ,$$

$$T_{12} = C_{2323}\,C_{1313} + C_{2323}\,C_{1212} + C_{1313}\,C_{1212} ;$$

A.11

$$\langle C'^{2}_{iiii} \rangle = \frac{1}{315}\left[35T_1 + 48T_2 + 12T_3 + 6T_4 + 20T_5 + 4T_6 + 8T_7 + 40T_8 + 8T_9 + 16T_{10} + 48T_{11} + 32T_{12}\right], \quad \text{A.12a}$$

$$\langle C'^{2}_{iijj} \rangle = \frac{1}{315}\left[3T_1 + 24T_2 + 39T_3 + 3T_4 + 12T_5 + 6T_6 + 0T_7 - 12T_8 + 30T_9 - 12T_{10} - 36T_{11} + 0T_{12}\right], \quad \text{A.12b}$$

$$\langle C'_{iijj}C'_{iikk} \rangle = \frac{1}{315}\left[T_1 + 0T_2 + 15T_3 + 3T_4 + 10T_5 + 14T_6 - 8T_7 - 4T_8 + 52T_9 - 28T_{10} - 12T_{11} + 16T_{12}\right], \quad \text{A.12c}$$

$$\langle C'_{iiij}C'_{ijjj} \rangle = \frac{1}{315}\left[3T_1 - 6T_2 - \frac{3}{2}T_3 - \frac{9}{2}T_4 + 3T_5 - 3T_6 - 6T_7 + 6T_8 + 0T_9 + 0T_{10} - 6T_{11} + 0T_{12}\right], \quad \text{A.12d}$$

$$\langle C'_{iiij}C'_{ijkk} \rangle = \frac{1}{315}\left[T_1 - 6T_2 + \frac{3}{2}T_3 - \frac{3}{2}T_4 + 4T_5 - 7T_6 + 4T_7 - T_8 - 2T_9 - T_{10} + 0T_{11} + 4T_{12}\right], \quad \text{A.12e}$$

$$\langle C'^{2}_{iiij} \rangle = \frac{1}{315}\left[5T_1 + 12T_2 + 3T_3 - 3T_4 - T_5 - 2T_6 - 4T_7 - 2T_8 - T_9 - 2T_{10} + 12T_{11} - 4T_{12}\right], \quad \text{A.12f}$$

$$\langle C'^{2}_{iijk} \rangle = \frac{1}{315}\left[T_1 + 12T_2 + 12T_3 + 0T_4 + T_5 - 4T_6 + 4T_7 - 4T_8 - 11T_9 + 8T_{10} - 12T_{11} - 8T_{12}\right], \quad \text{A.12g}$$

$$\langle C'^{2}_{ijij} \rangle = \frac{1}{315}\left[3T_1 + 27T_2 + 6T_3 + 3T_4 - 6T_5 + 0T_6 + 6T_7 + 6T_8 + 0T_9 - 6T_{10} - 6T_{11} + 18T_{12}\right], \quad \text{A.12h}$$

$$\langle C'^{2}_{ijik} \rangle = \frac{1}{315}\left[T_1 + 9T_2 + 3T_3 + 0T_4 - 2T_5 + 2T_6 - 2T_7 - T_8 - 2T_9 + 2T_{10} + 0T_{11} - 5T_{12}\right], \quad \text{A.12i}$$

$$\langle C'_{iiii} C'_{iijj} \rangle = \frac{1}{315} \left[5T_1 - 24T_2 + 12T_3 + 6T_4 + 26T_5 + 10T_6 \right.$$

$$\left. + 2T_7 - 2T_8 + 20T_9 + 16T_{10} + 12T_{11} - 16T_{12} \right], \quad \text{A.12j}$$

Applying A.8 to (36) and (37) of section 8.1 then

$$\langle A_i \rangle = \frac{1}{3} \left[\left\{ \langle C'^2_{i1} \rangle + \langle C'^2_{i2} \rangle + \langle C'^2_{i3} \rangle \right\} - \left\{ (C^0_{i1})^2 + (C^0_{i2})^2 + (C^0_{i3})^2 \right\} \right]$$

$$+ \left[\left\{ \langle C'^2_{i4} \rangle + \langle C'^2_{i5} \rangle + \langle C'^2_{i6} \rangle \right\} - \left\{ (C^0_{i4})^2 + (C^0_{i5})^2 + (C^0_{i6})^2 \right\} \right]$$

$$- \frac{1}{3} \left[\left\{ \langle C'_{i1} C'_{i2} \rangle + \langle C'_{i1} C'_{i3} \rangle + \langle C'_{i2} C'_{i3} \rangle \right\} - \left\{ C^0_{i1} C^0_{i2} + C^0_{i1} C^0_{i3} + C^0_{i2} C^0_{i3} \right\} \right],$$

and $\langle B_i \rangle$ is given by the same factors but with different numerical coefficients.

Using (A.1), (A.7a) to (A.7d), and (A.12a) to (A.12j) then

$$\left[\langle C'^2_{11} \rangle + \langle C'^2_{12} \rangle + \langle C'^2_{13} \rangle \right] - \left[(C^0_{11})^2 + (C^0_{12})^2 + (C^0_{13})^2 \right]$$

$$= \frac{1}{1575} \left[128T_1 + 312T_2 + 198T_3 - 94T_4 + 24T_5 - 116T_6 \right.$$

$$\left. - 72T_7 - 32T_8 - 164T_9 + 72T_{10} - 8T_{11} - 176T_{12} \right],$$

$$\left[\langle C'^2_{14} \rangle + \langle C'^2_{15} \rangle + \langle C'^2_{16} \rangle \right] - \left[(C^0_{14})^2 + (C^0_{15})^2 + (C^0_{16})^2 \right]$$

$$= \frac{1}{315} \left[11T_1 + 36T_2 + 18T_3 - 6T_4 - T_5 - 8T_6 \right.$$

$$\left. - 4T_7 - 8T_8 - 13T_9 + 4T_{10} + 12T_{11} - 16T_{12} \right],$$

$$\left[\langle C'_{11} C'_{12} \rangle + \langle C'_{11} C'_{13} \rangle + \langle C'_{12} C'_{13} \rangle \right] - \left[C^0_{11} C^0_{12} + C^0_{11} C^0_{13} + C^0_{12} C^0_{13} \right]$$

$$= \frac{1}{1575} \left[6T_1 - 156T_2 - 29T_3 - 23T_4 + 58T_5 - 82T_6 \right.$$

$$\left. + 36T_7 + 16T_8 + 12T_9 - 36T_{10} + 4T_{11} + 88T_{12} \right].$$

Hence,

$$\langle A_1 \rangle = \frac{1}{1575}\left[95\tfrac{2}{3}T_1 + 336T_2 + 165\tfrac{2}{3}T_3 - 53\tfrac{2}{3}T_4 - 16\tfrac{1}{3}T_5 \right.$$

$$\left. - 51\tfrac{1}{3}T_6 - 56T_7 - 56T_8 - 123\tfrac{2}{3}T_9 + 56T_{10} + 56T_{11} - 168T_{12} \right], \quad \text{A.13a}$$

and

$$\langle B_1 \rangle = \frac{1}{1575}\left[205\tfrac{1}{3}T_1 + 448T_2 + 298\tfrac{2}{3}T_3 - 149\tfrac{1}{3}T_4 + 56T_5 \right.$$

$$\left. - 224T_6 - 74\tfrac{2}{3}T_7 - 74\tfrac{2}{3}T_8 - 242\tfrac{2}{3}T_9 + 74\tfrac{2}{3}T_{10} + 74\tfrac{2}{3}T_{11} - 224T_{12} \right]. \quad \text{A.13b}$$

Similarly,

$$\left[\langle C'^2_{16}\rangle + \langle C'^2_{26}\rangle + \langle C'^2_{36}\rangle\right] - \left[(C^{\circ}_{16})^2 + (C^{\circ}_{26})^2 + (C^{\bullet}_{36})^2\right]$$

$$= \frac{1}{315}\left[11T_1 + 36T_2 + 18T_3 - 6T_4 - T_5 - 8T_6 - 4T_7 - 8T_8 \right.$$

$$\left. - 13T_9 + 4T_{10} + 12T_{11} - 16T_{12} \right],$$

$$\left[\langle C'^2_{46}\rangle + \langle C'^2_{56}\rangle + \langle C'^2_{66}\rangle\right] - \left[(C^{\circ}_{46})^2 + (C^{\circ}_{56})^2 + (C^{\bullet}_{66})^2\right]$$

$$= \frac{1}{1575}\left[18T_1 + 162T_2 + 53T_3 + T_4 - 36T_5 + 34T_6 - 32T_7 \right.$$

$$\left. - 22T_8 - 34T_9 + 32T_{10} + 12T_{11} - 86T_{12} \right],$$

and

$$\left[\langle C'_{16}C'_{26}\rangle + \langle C'_{16}C'_{36}\rangle + \langle C'_{26}C'_{36}\rangle\right] - \left[C^{\circ}_{16}C^{\circ}_{26} + C^{\circ}_{16}C^{\circ}_{36} + C^{\circ}_{26}C^{\bullet}_{36}\right]$$

$$= \frac{1}{315}\left[5T_1 - 18T_2 + \tfrac{3}{2}T_3 - \tfrac{15}{2}T_4 + 11T_5 - 17T_6 + 2T_7 + 4T_8 \right.$$

$$\left. - 4T_9 - 2T_{10} - 6T_{11} + 8T_{12} \right].$$

Hence,

$$\langle A_6 \rangle = \frac{1}{1575}\left[28T_1 + 252T_2 + 80\tfrac{1}{2}T_3 + 3\tfrac{1}{2}T_4 - 56T_5 + 49T_6 \right.$$

$$\left. - 42T_7 - 42T_8 - 49T_9 + 42T_{10} + 42T_{11} - 126T_{12} \right], \qquad \text{A.14a}$$

and

$$\langle B_6 \rangle = \frac{1}{1575}\left[95\tfrac{2}{3}T_1 + 336T_2 + 165\tfrac{2}{3}T_3 - 53\tfrac{2}{3}T_4 - 16\tfrac{1}{3}T_5 \right.$$

$$\left. - 51\tfrac{1}{3}T_6 - 56T_7 - 56T_8 - 123\tfrac{2}{3}T_9 + 56T_{10} + 56T_{11} - 168T_{12} \right] \qquad \text{A.14b}$$

$$= \langle A_1 \rangle .$$

These expressions can be further simplified by setting

$$a = (C_{44} + C_{55} + C_{66})^2 - 3(C_{44}C_{55} + C_{44}C_{66} + C_{55}C_{66}),$$

$$b = (C_{11} + C_{22} + C_{33})^2 - 3(C_{11}C_{22} + C_{11}C_{33} + C_{22}C_{33}),$$

$$c = (C_{12} + C_{13} + C_{23})^2 - 3(C_{12}C_{13} + C_{12}C_{23} + C_{13}C_{23}),$$

$$d = C_{11}(C_{12} + C_{13} - 2C_{23}) + C_{22}(C_{12} + C_{23} - 2C_{13})$$

$$+ C_{33}(C_{13} + C_{23} - 2C_{12}) . \qquad \text{A.15}$$

A.13 and A.14 then become

$$\langle A_1 \rangle = \langle B_6 \rangle = \frac{2P^2}{225} + \frac{1}{135}(24a + 7b + 13c + d),$$

$$\langle B_1 \rangle = \frac{8P^2}{675} + \frac{8}{135}(4a + 2b + 3c + d),$$

$$\langle A_6 \rangle = \frac{P^2}{150} + \frac{1}{90}(12a + b + 4c - 2d). \qquad \text{A.16}$$

BIBLIOGRAPHY:

[1] J.S. Koehler, "Imperfections in Nearly Perfect Crystals," John Wiley and Sons, New York, (1952), (p197).

[2] A. Granato and K. Lücke, J. App. Phys., 27, 583, 789, (1956).

[3] A. Seeger, Phil. Mag., 8th. Series, 1, 651, (1956).

[4] W. P. Mason and H. E. Bömmel, Jour. Acous. Soc. of Am., 28, 930, (1956).

[5] E. Lax, Phys. Rev., 115, 1591, (1959).

[6] A. B. Pippard, Phil. Mag., 46, 1104, (1955).

[7] M. S. Steinberg, Phys. Rev., 111, 425, (1958a).

[8] A. I. Akheizer et al, Soviet Physics, J.E.T.P., 5, 685, (1957).

[9] M. S. Steinberg, Phys. Rev., 109, 1486, (1958b).

[10] A. B. Bhatia, Phys. Rev., 97, 363, (1955).

[11] A. H. Wilson, "The Theory of Metals", The University Press, Cambridge, (1954).

[12] T. Holstein, Phys. Rev., 113, 479, (1959).

[13] N. F. Mott and H. Jones, "Properties of Metals and Alloys", Oxford University Press, London, (1936),(p88).

[14] D. H. Filson, Phys. Rev., 115, 1516, (1959).

[15] W. P. Mason and H. J. McSkimin, Jour. Acous. Soc. of Am., 19, 464, (1947).

[16] W. P. Mason and H. J. McSkimin, J. App. Phys., 19, 940, (1948).

[17] A. B. Bhatia, Jour. Acous. Soc. of Am., 31, 16, (1959).

[18] H. E. Bömmel, Phys. Rev., 96, 200, (1954).

B29787